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Cyclotomic valuation of q-Pochhammer symbols and q-integrality of basic hypergeometric series

by

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Abstract. We give a formula for the cyclotomic valuation of q-Pochhammer symbols in terms of (generalized) Dwork maps. We also obtain a criterion for the q-integrality of basic hypergeometric series in terms of certain step functions, which generalize Christol step functions. This provides suitable q-analogs of two results proved by Christol: a formula for the p-adic valuation of Pochhammer symbols and a criterion for the N-integrality of hypergeometric series.

1. Introduction. *Factorial ratios* form a remarkable class of sequences appearing regularly in combinatorics, number theory (e.g. [3, 7, 9, 21]), and mathematical physics and geometry (e.g. [5, 10, 12]). They are sequences of rational numbers of the form

$$Q_{e,f}(n) := \frac{(e_1 n)! \cdots (e_v n)!}{(f_1 n)! \cdots (f_w n)!}, \qquad n \geq 0,$$

where v and w are non-negative integers, and $e := (e_1, \ldots, e_v)$ and $f := (f_1, \ldots, f_w)$ are vectors whose coordinates are positive integers. Understanding how arithmetic properties of factorial ratios may depend on the integer parameters e_i and f_i leads to interesting and challenging problems. Landau [19] introduced the step function

$$\varDelta_{e,f}(x) := \sum_{i=1}^{v} \lfloor e_i x \rfloor - \sum_{j=1}^{w} \lfloor f_j x \rfloor \tag{1.1}$$

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and proved that the p-adic valuation of factorial ratios is given by

$$v_p(Q_{e,f}(n)) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\frac{n}{p^\ell}\right).$$

This result generalizes the classical Legendre formula: $v_p(n!) = \sum_{\ell=1}^{\infty} \lfloor n/p^\ell \rfloor$. Surprisingly, certain basic properties of the Landau function $\Delta_{e,f}$ turn out to characterize fundamental arithmetic properties of the corresponding factorial ratio and its generating series. Indeed, assuming for simplicity that $\sum_i e_i = \sum_j f_j$, we have the following results.

- (i) The sequence $(Q_{e,f}(n))_{n\geq 0}$ takes integer values if and only if $\Delta_{e,f}(x) \geq 0$, for all $x \in [0,1]$.
- (ii) The sequence $(Q_{e,f}(n))_{n\geq 0}$ has the p-Lucas property for all primes p (1) if and only if $\Delta_{e,f}(x) \geq 1$ for all $x \in [m_{e,f}, 1]$, where $m_{e,f} := 1/\max\{e_1, \dots, e_v, f_1, \dots, f_w\}$.
- (iii) The generating series of $(Q_{e,f}(n))_{n\geq 0}$ is algebraic(2) if and only if $\Delta_{e,f}(x) \in \{0,1\}$ for all $x \in [0,1]$.

Items (i) and (iii) were respectively proved by Landau [19] (see also [7]) and Rodriguez-Villegas [20] (as a consequence of [6]). Item (ii) corresponds to [2, Proposition 8.3] and was derived from [13, Theorem 3].

Choosing for example $e = (30, 1)$ and $f = (15, 10, 6)$, a straightforward computation shows that the corresponding sequence takes integer values, does not have the p-Lucas property for all primes, and has an algebraic generating series. At first sight, proving this result is not easy: for example, Rodriguez-Villegas [20] observed that the degree of algebraicity is 483 840.

These results have been generalized, replacing factorials by Pochhammer symbols and factorial ratios by hypergeometric sequences. We recall that the Pochhammer symbol $(x)_n$, also called the rising factorial, is defined as

$$(x)_n = x(x+1)\cdots(x+n-1),$$

if $n \geq 1$ and $(x)_0 = 1$, so that $(1)_n = n!$ and

$$(dn)! = d^{dn}\left(\frac{1}{d}\right)_n \cdots \left(\frac{d-1}{d}\right)_n (1)_n. \tag{1.2}$$

Given $\alpha \in \mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ and p a prime such that $v_p(\alpha) \geq 0$, Christol [11] provided

(1) That is $Q_{e,f}(pn+r) \equiv Q_{e,f}(n)Q_{e,f}(r) \bmod p$ for every $r \in \{0, \dots, p-1\}$ and $n \geq 0$.

(2) This means that the power series $\sum_{n=0}^{\infty} Q_{e,f}(n)x^n \in \mathbb{Q}[[x]]$ is algebraic over the field $\mathbb{Q}(x)$.

the following formula ([3]) for the p-adic valuation of Pochhammer symbols:

$$v_p((\alpha)_n) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{n - \lfloor 1-\alpha \rfloor}{p^\ell} - D_p^\ell(\alpha) + 1 \right\rfloor, \tag{1.3}$$

where $D_p(\alpha)$ is defined as the unique rational number whose denominator is not divisible by p and such that $pD_p(\alpha) - \alpha \in \{0, \dots, p-1\}$. The maps $\alpha \mapsto D_p(\alpha)$ were first introduced by Dwork [15] and are now referred to as *Dwork maps*. When $\alpha = 1$, we have $D_p(1) = 1$ and we retrieve Legendre's formula. Note also that if $v_p(\alpha) < 0$, then simply $v_p((\alpha)_n) = nv_p(\alpha)$.

Given two vectors $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_v)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_w)$ with coordinates in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we define the (generalized) *hypergeometric sequence*

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n) := \frac{(\alpha_1)_n \cdots (\alpha_v)_n}{(\beta_1)_n \cdots (\beta_w)_n} \in \mathbb{Q}, \qquad n \geq 0. \tag{1.4}$$

The above restriction on the rational parameters β_j ensures that $Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ is well-defined for all $n \geq 0$. We also assume that the parameters α_i do not belong to $\mathbb{Z}_{\leq 0}$, since otherwise $Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ would vanish for all n large enough, which would make them irrelevant for our purpose. These sequences and their generating series have attracted a lot of attention since the time of Gauss. According to (1.2), the study of factorial ratios reduces to the study of certain hypergeometric sequences. Again, understanding how the arithmetic properties of hypergeometric sequences may depend on the rational parameters α_i and β_j leads to fascinating questions.

We let $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ denote the least common multiple of the denominators of the parameters α_i and β_j. In [11], Christol introduced new step functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \cdot)$, for every $a \in \{1, \dots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, which play the same role for hypergeometric sequences as the Landau function $\Delta_{e,f}$ does for factorial ratios. We refer the reader to Section 5.1 for a definition.

Analogs of (i)–(iii) have been respectively obtained by Christol [11], Adamczewski, Bell, and Delaygue [2], and Beukers and Heckman [6] ([4]). We point out that, for the analog of (i), it is more natural to consider N-integrality of the sequence $(Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n))_{n\geq 0}$, that is, to ask whether there exists a non-zero integer a such that $a^n Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n) \in \mathbb{Z}$ for all $n \geq 0$. Also, for the analog of (ii), it is more natural to consider the p-Lucas property for all but finitely many primes in a given residue class modulo $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Finally, the required conditions about the Landau function must now be satisfied by the

([3]) More exactly, formula (1.3) is a reformulation with floor functions of Christol's result, as given in [14, Section 5.3].

([4]) We refer the reader to [11, 2, 6] for precise statements. The reformulation in terms of Christol step functions of the famous interlacing criterion of Beukers and Heckmann can be found in [14].

Christol functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$ for all $a \in \{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. In particular, the analog of (i) proved by Christol [11] reads as follows.

THEOREM A. *Let* $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_u)$ *and* $\boldsymbol{\beta} := (\beta_1,\dots,\beta_v)$ *be two vectors whose coordinates belong to* $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. *Then the following two assertions are equivalent:*

(a) *The hypergeometric sequence* $(Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n))_{n\geq 0}$ *is* N*-integral.*
(b) *For every* a *in* $\{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ *coprime to* $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *and all* x *in* $\mathbb{R}$, *we have* $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x) \geq 0$.

A remarkable feature of (i)–(iii) and of the results proved in [11, 2, 6] is that they provide simple algorithms, given in terms of suitable step functions, that allow one to decide whether certain fundamental arithmetic properties of factorial ratios and hypergeometric sequences hold ([5]).

1.1. Main results. In this paper, our main objective is to prove q-analogs of formula (1.3) and Theorem A. From now on, we let q denote a fixed transcendental complex number.

We are going to define suitable q-analogs of the Pochhammer symbol $(\alpha)_n$ and of the hypergeometric term $Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$, which belong to the field $\mathbb{Q}(q)$. In this framework, the p-adic valuations are replaced by the cyclotomic valuations, while the notion of N-integrality is replaced by q-integrality. For every positive integer b, we let $\phi_b(q) \in \mathbb{Z}[q]$ denote the bth cyclotomic polynomial and v_{ϕ_b} stands for the valuation of $\mathbb{Q}(q)$ associated with $\phi_b(q)$ (see Section 2.1 for a definition). A sequence $(R(q;n))_{n\geq 0}$ with values in $\mathbb{Q}(q)$ and first term $R(q;0)=1$ is said to be *q-integral* if there exists $C(q) \in \mathbb{Z}[q]\setminus\{0\}$ such that $C(q)^n R(q;n) \in \mathbb{Z}[q]$ for all $n \geq 0$.

For every positive integer n, the q-analog of the integer n is defined as $[n]_q = 1 + q + \cdots + q^{n-1}$, while $[0]_q = 0$. It is actually convenient to write

$$[n]_q = \frac{1-q^n}{1-q},$$

while keeping in mind that this ratio belongs to $\mathbb{Z}[q]$. It follows that

$$[n]_q = \prod_{b\geq 2,\, b\mid n} \phi_b(q),$$

which specializes as

$$n = \prod_{b\geq 2,\, b\mid n} \phi_b(1). \tag{1.5}$$

([5]) We also refer the reader to [4] for more general results about integrality of A-hypergeometric series.

We recall that $\phi_b(1) = 1$ if b is divisible by at least two distinct primes, while $\phi_{p^\ell}(1) = p$ when p is a prime and ℓ is a positive integer. We deduce that

$$v_p(n) = \sum_{\ell=1}^{\infty} v_{\phi_{p^\ell}}([n]_q). \tag{1.6}$$

This formula shows that, in some sense, the arithmetic of q-analogs is finer than that of integers. The q-analog of $n!$ is defined as

$$[n]!_q := \prod_{i=1}^{n} \frac{1-q^i}{1-q}\,.$$

Given $\alpha = r/s$ a rational number, the q-analog of the Pochhammer symbol $(\alpha)_n$ is most often defined as (see, for instance, [16])

$$\frac{(q^\alpha;q)_n}{(1-q)^n} \in \mathbb{Q}(q^{1/s}),$$

where we let $(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i)$ denote the q-Pochhammer symbol (also called the q-shifted factorial). Substituting q by q^s, we obtain a slightly different q-analog of $(\alpha)_n$:

$$\frac{(q^r;q^s)_n}{(1-q^s)^n} \in \mathbb{Q}(q). \tag{1.7}$$

We note that

$$\lim_{q\to 1} \frac{(q^\alpha;q)_n}{(1-q)^n} = \lim_{q\to 1} \frac{(q^r;q^s)_n}{(1-q^s)^n} = (\alpha)_n.$$

The latter has several advantages which are discussed in Section 2. In the end, it is sufficient for our discussion to consider q-Pochhammer symbols of the form

$$(q^r;q^s)_n := \prod_{i=0}^{n-1}(1-q^{r+si}) \in \mathbb{Z}[q^{-1},q],$$

where r and s are two integers, $s \neq 0$. This product is non-zero if and only if $r/s \notin \mathbb{Z}_{\leq 0}$ or $n \leq -r/s$. The usual extension to negative arguments n is given by

$$(q^r;q^s)_n = \prod_{i=1}^{-n} \frac{1}{(1-q^{r-is})} = \frac{1}{(q^{r-s};q^{-s})_{-n}}, \tag{1.8}$$

which is well-defined if and only if $r/s \notin \mathbb{Z}_{>0}$ or $n > -r/s$.

Our first main result, which provides a q-analog of formula (1.3) as well as its extension to negative arguments, involves a generalization of Dwork maps where the prime number p is replaced by an arbitrary positive integer b. Given a positive integer b and a rational number α whose denominator is coprime to b, we show in Section 3.1 that there exists a unique rational number $D_b(\alpha)$ whose denominator is coprime to b and which satisfies $bD_b(\alpha) - \alpha \in \{0, \ldots, b-1\}$. When $b = p$ is prime, we retrieve the classical Dwork map D_p.

THEOREM 1.1. *Let r and s be two integers, $s \neq 0$, and $\alpha := r/s$. Let b be a positive integer, $c := \gcd(r, s, b)$, $b' := b/c$, and $s' := s/c$. Let $n \in \mathbb{Z}$ be such that $(q^r; q^s)_n$ is well-defined and non-zero. Then*

$$v_{\phi_b}((q^r; q^s)_n) = \begin{cases} \lfloor cn/b - D_{b'}(\alpha) - \lfloor 1-\alpha \rfloor / b' \rfloor + 1 & \text{if } \gcd(s', b') = 1, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 1.2. Recall that $v_{\phi_b}((1-q^s)^n) = nv_{\phi_b}(1-q^s)$ and $v_{\phi_b}(1-q^s) = 1$ if b divides s and 0 otherwise. Hence we can easily derive from Theorem 1.1 a formula for the ϕ_b-valuation of the q-analog of $(\alpha)_n$ given in (1.7).

We now define q-analogs of hypergeometric sequences with rational parameters. For $i \in \{1, \ldots, v\}$ and $j \in \{1, \ldots, w\}$, we let (r_i, s_i) and (t_j, u_j) be pairs of integers such that $s_i \neq 0$ and $u_j \neq 0$. We set

$$\mathbf{r} := ((r_1, s_1), \ldots, (r_v, s_v)) \quad \text{and} \quad \mathbf{t} := ((t_1, u_1), \ldots, (t_w, u_w)),$$

together with $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_v)$ and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_w)$, where $\alpha_i := r_i/s_i$ and $\beta_j := t_j/u_j$. Let $d_{\mathbf{r},\mathbf{t}} := \operatorname{lcm}(s_1, \ldots, s_v, u_1, \ldots, u_w)$. With this notation, we define the *q-hypergeometric sequence*

$$Q_{\mathbf{r},\mathbf{t}}(q; n) := \frac{(q^{r_1}; q^{s_1})_n \cdots (q^{r_v}; q^{s_v})_n}{(q^{t_1}; q^{u_1})_n \cdots (q^{t_w}; q^{u_w})_n}, \qquad n \geq 0. \tag{1.9}$$

Note that, similarly to (1.4), $Q_{\mathbf{r},\mathbf{t}}(q; n)$ is well-defined for all $n \geq 0$ when the rational numbers β_j do not belong to $\mathbb{Z}_{\leq 0}$. In addition, we assume that the rational numbers α_i do not belong to $\mathbb{Z}_{\leq 0}$, since otherwise $Q_{\mathbf{r},\mathbf{t}}(q; n)$ would vanish for all n large enough, which would make them irrelevant for our purpose.

Our second main result is a q-analog of Theorem A. It involves new step functions $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$, $b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$, which generalize Christol step functions. They are introduced in Section 5, where we also show that $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \cdot)$ for b coprime to $d_{\mathbf{r},\mathbf{t}}$ and $ba \equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$. Thus, we only define new functions for b not coprime to $d_{\mathbf{r},\mathbf{t}}$. The appearance of these new functions makes the proof of Theorem 1.3 substantially more tricky than that of Theorem C.

THEOREM 1.3. *Keeping the previous notation and assumptions, assume also that $s_1, \ldots, s_v$ are positive. Then the following two assertions are equivalent:*

(i) *The sequence $(Q_{\mathbf{r},\mathbf{t}}(q; n))_{n \geq 0}$ is q-integral.*
(ii) *For every $b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$ and all x in $\mathbb{R}$, we have $\Xi_{\mathbf{r},\mathbf{t}}(b, x) \geq 0$.*

A generalization of Theorem 1.3 with no restriction on the parameters $s_1, \dots, s_v \in \mathbb{Z} \setminus \{0\}$ is obtained as Theorem 5.6 in Section 5.4.

Given some parameters $\mathbf{r}$ and $\mathbf{t}$, checking whether (ii) is satisfied is a simple exercise. Indeed, for every b in $\{1, \dots, d_{\mathbf{r},\mathbf{t}}\}$, the step function $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$ is non-negative on $\mathbb{R}$ if and only if $\Xi_{\mathbf{r},\mathbf{t}}(b, \Gamma_i) \geq 0$ for a finite number of points Γ_i which are explicitly given in (5.6). As mentioned in Remark 5.5, one can easily compute $\Xi_{\mathbf{r},\mathbf{t}}(b, \Gamma_i)$. In particular, the proof of Theorem 1.3 leads to an algorithm which, given $\mathbf{r}$ and $\mathbf{t}$, decides whether $Q_{\mathbf{r},\mathbf{t}}(q; n)$ is q-integral or not.

REMARK 1.4. Strictly speaking, $Q_{\mathbf{r},\mathbf{t}}(q; n)$ is not a q-analog of the hypergeometric term $Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$. Instead, (1.7) shows that a suitable q-analog can be defined as

$$Q'_{\mathbf{r},\mathbf{t}}(q; n) := \left(\frac{\prod_{j=1}^{w} (1 - q^{u_j})}{\prod_{i=1}^{v} (1 - q^{s_i})} \right)^n Q_{\mathbf{r},\mathbf{t}}(q; n).$$

Indeed, we have

$$\lim_{q \to 1} Q'_{\mathbf{r},\mathbf{t}}(q; n) = Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n). \tag{1.10}$$

Since the q-integrality of $(Q_{\mathbf{r},\mathbf{t}}(q; n))_{n \geq 0}$ is equivalent to that of $(Q'_{\mathbf{r},\mathbf{t}}(q; n))_{n \geq 0}$, we find it more convenient to work with the simpler expression $Q_{\mathbf{r},\mathbf{t}}(q; n)$.

We infer from (1.10) that the q-integrality of the sequence $(Q_{\mathbf{r},\mathbf{t}}(q; n))_{n \geq 0}$ implies the N-integrality of $(Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n))_{n \geq 0}$. This is consistent with Theorems 1.3 and C since $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot) = \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \cdot)$ when $ba \equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$. However, the converse does not always hold, depending on the behaviour of $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$ for b not coprime to $d_{\mathbf{r},\mathbf{t}}$.

For example, let us consider the vectors

$$\mathbf{r} := ((1, 3), (2, 3)) \quad \text{and} \quad \mathbf{t} := ((1, 2), (1, 1)).$$

Then $\boldsymbol{\alpha} = (1/3, 2/3)$ and $\boldsymbol{\beta} = (1/2, 1)$. We deduce from (1.2) (or from Theorem A) that the hypergeometric sequence

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n) = \frac{(1/3)_n (2/3)_n}{(1/2)_n (1)_n}, \qquad n \geq 0,$$

is N-integral. However, $\Xi_{\mathbf{r},\mathbf{t}}(3, 1/2) < 0$ (see Section 6.1 for more details) and thus, according to Theorem 1.3, the q-hypergeometric sequence

$$Q_{\mathbf{r},\mathbf{t}}(q; n) = \frac{(q; q^3)_n (q^2; q^3)_n}{(q; q^2)_n (q; q)_n}, \qquad n \geq 0,$$

fails to be q-integral.

1.2. Organization of the paper. In Section 2, we discuss our choice for the q-analog of the Pochhammer symbol $(\alpha)_n$ and show how to relate our results on q-hypergeometric sequences to basic hypergeometric series as they are usually defined. In Section 3, we extend the definition of Dwork maps to arbitrary integers b and prove some of their basic properties. We also prove Theorem 1.1, as well as a formula for the cyclotomic valuation of q-hypergeometric terms. The latter is given in terms of certain step functions $\Delta_b^{\mathbf{r},\mathbf{t}}$, which are introduced in this section. In Section 4, we deduce a first criterion for the q-integrality of q-hypergeometric sequences, which depends on the behaviour of $\Delta_b^{\mathbf{r},\mathbf{t}}$ for all but finitely many integers b. We also discuss the extension of this result to negative arguments n. These first criteria for q-integrality are not very satisfactory because they imply checking certain properties of an infinite number of step functions. We fill this gap in Section 5, where we introduce the finitely many step functions $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$, $b\in\{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$, and prove Theorem 1.3. Finally, we provide some illustrations of Theorem 1.3 in Section 6.

2. Choices for the q-analogs of Pochhammer symbols and hypergeometric functions. The notion of q-analog is loosely defined: for $a(q)$ to be a q-analog of a term a, one only requires that $a(q)$ tends to a as q tends to 1. While everyone agrees on the definition of $[n]_q$ and $[n]!_q$, this requires a fair amount of choice for more general expressions. Depending on the nature of the properties one wishes to study, one may have to make one choice rather than another. In this section, we discuss in more detail our own choices for the q-analogs of Pochhammer symbols and hypergeometric series, as well as how our results translate when considering other natural q-analogs.

2.1. Cyclotomic valuations and q-valuation. We recall that, for every positive integer b, $\phi_b(q)\in\mathbb{Z}[q]$ stands for the bth cyclotomic polynomial. It is well-known that $\phi_b(q)$ is irreducible over $\mathbb{Z}[q]$. If R and S belong to $\mathbb{Z}[q]\setminus\{0\}$, then we let $v_{\phi_b}(R)$ denote the ϕ_b-valuation of R, that is, the largest non-negative integer ν such that $\phi_b(q)^\nu$ divides R. We also set $v_{\phi_b}(0):=+\infty$. The ϕ_b-valuation extends naturally to $\mathbb{Q}(q)$ by setting $v_{\phi_b}(R/S):=v_{\phi_b}(R)-v_{\phi_b}(S)$.

We also let v_q denote the valuation of $\mathbb{Q}(q)$ which is associated with the irreducible polynomial q in the same way.

2.2. q-Analogs of Pochhammer symbols. We explain now why we prefer to choose

$$\frac{(q^r;q^s)_n}{(1-q^s)^n}\in\mathbb{Q}(q) \tag{2.1}$$

as a q-analog of the Pochhammer symbol $(\alpha)_n$, $\alpha = r/s$, instead of the more standard

$$\frac{(q^\alpha;q)_n}{(1-q)^n} \in \mathbb{Q}(q^{1/s}). \tag{2.2}$$

There are three main reasons for our preference. The first one, which was already mentioned in the introduction, is that we find it more natural to work in the field $\mathbb{Q}(q)$ instead of working in the field $\bigcup_{s\geq 1}\mathbb{Q}(q^{1/s})$ and dealing with non-integer powers of q. The second one is that it offers more flexibility. For example,

$$\frac{(q;q^2)_n}{(1-q^2)^n}, \quad \frac{(q^3;q^6)_n}{(1-q^6)^n}, \quad \text{and} \quad \frac{(q^{-1};q^{-2})_n}{(1-q^{-2})^n}$$

provide three different q-analogs of $(1/2)_n$. The third one comes from the useful equality (1.2), which we recall here for the reader's convenience:

$$(dn)! = d^{dn}\left(\frac{1}{d}\right)_n\left(\frac{2}{d}\right)_n\cdots\left(\frac{d-1}{d}\right)_n(1)_n. \tag{2.3}$$

With the choice of $(q^\alpha;q)_n/(1-q)^n$, we do not obtain a nice q-deformation of (2.3). Indeed, take for instance $d=2$, so that

$$(2n)! = 4^n(1/2)_n(1)_n.$$

The q-analog of the left-hand side of (2.3) is

$$\frac{(q;q)_{2n}}{(1-q)^{2n}} = \frac{(q;q^2)_n(q^2;q^2)_n}{(1-q)^{2n}} = (-q^{1/2};q)_n(-q;q)_n\frac{(q^{1/2};q)_n}{(1-q)^n}\frac{(q;q)_n}{(1-q)^n},$$

therefore introducing minus signs in q-Pochhammer symbols. In contrast, the choice $(q^r;q^s)_n/(1-q^s)^n$ ensures the following nice q-deformation of (2.3):

$$[dn]!_q = \prod_{i=1}^{dn}\frac{1-q^i}{1-q} = \left(\frac{1-q^d}{1-q}\right)^{dn}\prod_{i=1}^{d}\frac{(q^i;q^d)_n}{(1-q^d)^n}.$$

Remark 2.1. Let d be a positive integer. Since q is transcendental over $\mathbb{Q}$, there is an isomorphism of $\mathbb{Z}$-modules given by

$$\varphi : \mathbb{Z}[q^{1/d}] \to \mathbb{Z}[q], \qquad P(q^{1/d}) \mapsto P(q).$$

In particular, $\mathbb{Z}[q^{1/d}]$ is a Euclidean ring whose irreducible elements are of the form $P(q^{1/d})$ where $P(q)$ is an irreducible polynomial in $\mathbb{Z}[q]$. The isomorphism φ extends to an isomorphism between the rings of Laurent polynomials $\mathbb{Z}[q^{-1/d},q^{1/d}]$ and $\mathbb{Z}[q^{-1},q]$, as well as between the fields $\mathbb{Q}(q^{1/d})$ and $\mathbb{Q}(q)$. In particular, if we let $v_{b,s}$ denote the valuation in $\mathbb{Q}(q^{1/s})$ associated with the irreducible polynomial $\phi_b(q^{1/s}) \in \mathbb{Z}[q^{1/s}]$ and if we take $\alpha = r/s$, then we obtain

$$v_{b,s}((q^\alpha;q)_n/(1-q)^n) = v_{\phi_b}((q^r;q^s)_n/(1-q^s)^n).$$

This shows that there is no loss of generality in choosing (2.1) as a q-analog of $(\alpha)_n$.

2.3. q-Analogs of generalized hypergeometric series. Let us first recall the standard notation for hypergeometric series (with rational parameters). With two vectors $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_v)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_w)$ whose coordinates belong to $\mathbb{Q} \setminus \mathbb{Z}_{\le 0}$, we associate the generalized hypergeometric series defined by

$$ {}_vF_w\left(\begin{matrix} \alpha_1, & \dots, & \alpha_v \\ \beta_1, & \dots, & \beta_w \end{matrix}\,\middle|\, x\right) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_v)_n}{(\beta_1)_n \cdots (\beta_w)_n n!} x^n, $$

while we usually prefer to work with its companion power series

$$ F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x) := {}_{v+1}F_w\left(\begin{matrix} \alpha_1, & \dots, & \alpha_v, & 1 \\ \beta_1, & \dots, & \beta_w & \end{matrix}\,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_v)_n}{(\beta_1)_n \cdots (\beta_w)_n} x^n. $$

The *basic hypergeometric series* is defined as

$$ {}_v\phi_w\left(\begin{matrix} q^{\alpha_1}, & \dots, & q^{\alpha_v} \\ q^{\beta_1}, & \dots, & q^{\beta_w} \end{matrix}\,\middle|\, q; x\right) := \sum_{n=0}^{\infty} \frac{(q^{\alpha_1}; q)_n \cdots (q^{\alpha_v}; q)_n}{(q^{\beta_1}; q)_n \cdots (q^{\beta_w}; q)_n (q; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+w-v} x^n. $$

It is a generalization of the classical ${}_2\phi_1$ introduced by Heine [17] and the most standard q-analog of the hypergeometric series ${}_vF_w$ (see, for instance, the monograph [16] for more on this topic). In fact, it is a q-analog up to renormalization by a factor $(q-1)^{(w-v)n}$, that is,

$$ \lim_{q\to 1} {}_{v+1}\phi_w\left(\begin{matrix} q^{\alpha_1}, & \dots, & q^{\alpha_v}, & q \\ q^{\beta_1}, & \dots, & q^{\beta_w} & \end{matrix}\,\middle|\, q; (q-1)^{w-v} x\right) = F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x). $$

Hence a first q-analog of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$ is given by

$$ \begin{aligned} F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q; x) &:= {}_{v+1}\phi_w\left(\begin{matrix} q^{\alpha_1}, & \dots, & q^{\alpha_v}, & q \\ q^{\beta_1}, & \dots, & q^{\beta_w} & \end{matrix}\,\middle|\, q; (q-1)^{w-v} x\right) \\ &= \sum_{n=0}^{\infty} \frac{(q^{\alpha_1}; q)_n \cdots (q^{\alpha_v}; q)_n}{(q^{\beta_1}; q)_n \cdots (q^{\beta_w}; q)_n} \cdot (1-q)^{(w-v)n} q^{(w-v)\binom{n}{2}} x^n. \end{aligned} \tag{2.4} $$

Now, choosing $(q^\alpha; q)_n/(1-q)^n$ as a q-analog of the Pochhammer symbol $(\alpha)_n$, we obtain another natural q-analog of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$, namely

$$ F^{(2)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q; x) := \sum_{n=0}^{\infty} \frac{(q^{\alpha_1}; q)_n \cdots (q^{\alpha_v}; q)_n}{(q^{\beta_1}; q)_n \cdots (q^{\beta_w}; q)_n} \cdot (1-q)^{(w-v)n} x^n. \tag{2.5} $$

The two definitions only differ by the factor $q^{(w-v)\binom{n}{2}}$. In particular, they coincide when $v = w$.

Finally, choosing $(q^r; q^s)_n/(1-q^s)^n$ as a q-analog of the Pochhammer symbol $(\alpha)_n$, $\alpha = r/s$, we obtain a third natural q-analog of $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$:
(2.6)

$$F_{\boldsymbol{r},\boldsymbol{t}}(q;x) := \sum_{n=0}^{\infty} \frac{(q^{r_1};q^{s_1})_n \cdots (q^{r_v};q^{s_v})_n}{(q^{t_1};q^{u_1})_n \cdots (q^{t_w};q^{u_w})_n} \cdot \left(\frac{(1-q^{u_1})\cdots(1-q^{u_w})}{(1-q^{s_1})\cdots(1-q^{s_v})} \right)^n x^n,$$

where $\boldsymbol{r} = ((r_1,s_1),\dots,(r_v,s_v))$, $\boldsymbol{t} = ((t_1,u_1),\dots,(t_w,u_w))$, and $r_i/s_i = \alpha_i$ and $t_j/u_j = \beta_j$ for all i and j.

Thus, we have three different natural q-analogs of the generalized hypergeometric series $F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$. We observe that both $F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x)$ and $F^{(2)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x)$ have coefficients in $\mathbb{Q}(q^{1/d})$, where $d = d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is the least common multiple of the denominators of the rational numbers α_i and β_j. In contrast, $F_{\boldsymbol{r},\boldsymbol{t}}(q;x)$ has coefficients in $\mathbb{Q}(q)$ and there exist infinitely many vectors $\boldsymbol{r}$ and $\boldsymbol{t}$ such that

$$\lim_{q\to 1} F_{\boldsymbol{r},\boldsymbol{t}}(q;x) = F_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x).$$

Indeed, if $\boldsymbol{r} = ((r_1,s_1),\dots,(r_v,s_v))$ and $\boldsymbol{t} = ((t_1,u_1),\dots,(t_w,u_w))$ is such a pair of vectors, then for each pair (a,b) occurring either in $\boldsymbol{r}$ or in $\boldsymbol{t}$, we can choose a non-zero integer k and replace (a,b) by (ka,kb).

2.4. q-Integrality and $q^{1/d}$-integrality for basic hypergeometric series. A power series $F(q;x) \in 1 + x\mathbb{Q}(q)[[x]]$ is said to be *q-integral* if the sequence formed by its coefficients is q-integral, or, in other words, if there exists $C(q) \in \mathbb{Z}[q] \setminus \{0\}$ such that $F(q;C(q)x) \in \mathbb{Z}[q][[x]]$.

Similarly, we say that a power series $F(q;x) \in 1 + x\mathbb{Q}(q^{1/d})[[x]]$ is *$q^{1/d}$-integral* if there exists $C(q) \in \mathbb{Z}[q^{1/d}]\setminus\{0\}$ such that $F(q;C(q)x) \in \mathbb{Z}[q^{1/d}][[x]]$. According to Remark 2.1, $F(q;x)$ is $q^{1/d}$-integral if and only if $F(q^d;x)$ is q-integral.

Now, we show how Theorem 1.3 can be used to study the $q^{1/d}$-integrality of $F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x)$ and $F^{(2)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x)$, as well as the q-integrality of $F_{\boldsymbol{r},\boldsymbol{t}}(q;x)$. Recall that

$$F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q^d;x) = \sum_{n=0}^{\infty} \frac{(q^{d\alpha_1};q^d)_n \cdots (q^{d\alpha_v};q^d)_n}{(q^{d\beta_1};q^d)_n \cdots (q^{d\beta_w};q^d)_n} \cdot (1-q^d)^{(w-v)n} q^{d(w-v)\binom{n}{2}} x^n,$$

$$F^{(2)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q^d;x) = \sum_{n=0}^{\infty} \frac{(q^{d\alpha_1};q^d)_n \cdots (q^{d\alpha_v};q^d)_n}{(q^{d\beta_1};q^d)_n \cdots (q^{d\beta_w};q^d)_n} \cdot (1-q^d)^{(w-v)n} x^n.$$

Setting $\boldsymbol{r} := ((d\alpha_1,d),\dots,(d\alpha_v,d))$ and $\boldsymbol{t} := ((d\beta_1,d),\dots,(d\beta_w,d))$, we obtain

$$F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q^d;x) = \sum_{n=0}^{\infty} Q_{\mathbf{r},\mathbf{t}}(q;n)(1-q^d)^{(w-v)n} q^{d(w-v)\binom{n}{2}} x^n,$$
$$F^{(2)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q^d;x) = \sum_{n=0}^{\infty} Q_{\mathbf{r},\mathbf{t}}(q;n)(1-q^d)^{(w-v)n} x^n,$$
$$F_{\boldsymbol{r},\boldsymbol{t}}(q;x) = \sum_{n=0}^{\infty} Q_{\mathbf{r},\mathbf{t}}(q;n)\left(\frac{(1-q^{u_1})\cdots(1-q^{u_w})}{(1-q^{s_1})\cdots(1-q^{s_v})}\right)^n x^n.$$

Note that for q-integrality, we can omit factors of the form $h(q)^n$ with $h(q) \in \mathbb{Q}(q)$ such as

$$(1-q^d)^{(w-v)n} \quad \text{and} \quad \left(\frac{(1-q^{u_1})\cdots(1-q^{u_w})}{(1-q^{s_1})\cdots(1-q^{s_v})}\right)^n.$$

It follows that $F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x)$ is $q^{1/d}$-integral if and only if $Q_{\mathbf{r},\mathbf{t}}(q;n)$ is q-integral and

$$v_q\big(q^{d(w-v)\binom{n}{2}}\big) \geq an \qquad \forall n \geq 0,$$

for some integer a, that is,

$$F^{(1)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x) \text{ is } q^{1/d}\text{-integral} \iff (Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0} \text{ is } q\text{-integral and } w \geq v.$$

We also deduce that

$$\begin{aligned} F^{(2)}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;x) \text{ is } q^{1/d}\text{-integral} &\iff F_{\boldsymbol{r},\boldsymbol{t}}(q;x) \text{ is } q\text{-integral} \\ &\iff (Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0} \text{ is } q\text{-integral.} \end{aligned}$$

2.5. Irreducible factors of q-Pochhammer symbols and q-integrality of q-hypergeometric sequences. Throughout this paper, we work only with ratios of products of terms of the form $(q^r;q^s)_n$ and $1-q^s$, where r and s are integers, $s \neq 0$, and n is an integer.

Let us first recall that, for every positive integer a, we have

$$1-q^a = -\prod_{b\mid a}\phi_b(q) \quad \text{and} \quad 1-q^{-a} = -q^{-a}(1-q^a) = q^{-a}\prod_{b\mid a}\phi_b(q). \tag{2.7}$$

Let $n \in \mathbb{Z}$. It follows that any ratio of products of terms of the form $(q^r;q^s)_n$ and $1-q^s$, where r and s are integers and $s \neq 0$, has a unique decomposition of the form

$$\pm q^{v_{q,n}} \prod_{b=1}^{\infty} \phi_b(q)^{v_{b,n}}, \tag{2.8}$$

where $v_{q,n}, v_{1,n}, \ldots$ are integers and $v_{b,n} = 0$ for all but finitely many positive integers b. The integer $v_{q,n}$ is the q-valuation of this ratio and, for every $b \geq 1$, the integer $v_{b,n}$ is its ϕ_b-valuation.

Remark 2.2. A term of the form (2.8) belongs to $\mathbb{Z}[q]$ if and only if the integers $v_{q,n}, v_{1,n}, v_{2,n}, \ldots$ are all non-negative. When only the integers $v_{1,n}, v_{2,n}, \ldots$ are non-negative, then the term belongs to $\mathbb{Z}[q^{-1}, q]$.

2.5.1. *The q-valuation of q-Pochhammer symbols.* Let n be a positive integer and r and s be two integers, $s \neq 0$. Let us assume that $(q^r; q^s)_n$ is well-defined and non-zero. We let $\mathcal{N} := \{i \in \{0, \ldots, n-1\} : r + is < 0\}$. Then

$$v_q((q^r; q^s)_n) = \sum_{i \in \mathcal{N}} (r + is).$$

We deduce the following results:

(i) If r and s are non-negative, then $v_q((q^r; q^s)_n) = 0$.
(ii) If r is negative and s positive, then the sequence $(v_q((q^r; q^s)_n))_{n \geq 0}$ remains bounded.
(iii) If s is negative, then

$$v_q((q^r; q^s)_n) \underset{n \to +\infty}{\sim} s\binom{n}{2}. \tag{2.9}$$

Now, let n be a negative integer. We can derive similar results from the expression

$$(q^r; q^s)_n = \frac{1}{(q^{r-s}; q^{-s})_{-n}} \cdot$$

In particular, $(v_q((q^r; q^s)_n))_{n \leq 0}$ remains bounded if s is negative, and

$$v_q((q^r; q^s)_n) \underset{n \to -\infty}{\sim} s\binom{-n}{2} \tag{2.10}$$

if s is positive.

2.5.2. *Asymptotics for cyclotomic and q-valuations of q-hypergeometric terms.* Let us consider the q-hypergeometric sequence

$$Q_{\mathbf{r},\mathbf{t}}(q; n) = \frac{(q^{r_1}; q^{s_1})_n \cdots (q^{r_v}; q^{s_v})_n}{(q^{t_1}; q^{u_1})_n \cdots (q^{t_w}; q^{u_w})_n}, \qquad n \geq 0,$$

which we assume to be well-defined and not eventually zero. We first infer from (2.7) that

$$v_{\phi_b}(Q_{\mathbf{r},\mathbf{t}}(q; n)) = O(n) \tag{2.11}$$

for every positive integer b. Let $\mathcal{N}_1 := \{i \in \{1, \ldots, v\} : s_i < 0\}$, $\mathcal{N}_2 := \{j \in \{1, \ldots, w\} : u_j < 0\}$, and $s = \sum_{i \in \mathcal{N}_1} s_i - \sum_{j \in \mathcal{N}_2} u_j$. Using (i)–(iii) above, we deduce that

$$v_q(Q_{\mathbf{r},\mathbf{t}}(q; n)) = s\binom{n}{2} + O(n). \tag{2.12}$$

It follows from (2.8), Remark 2.2, and equalities (2.11) and (2.12) that

$$(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0} \text{ is } q\text{-integral} \iff s\geq 0 \text{ and } v_{\phi_b}(Q_{\mathbf{r},\mathbf{t}}(q;n))\geq 0\ \forall b\gg 1 \tag{2.13}$$

and

$$\exists C(q)\in\mathbb{Z}[q]\setminus\{0\}\ \forall n\geq 0,\ C(q)^n Q_{\mathbf{r},\mathbf{t}}(q;n)\in\mathbb{Z}[q^{-1},q] \iff v_{\phi_b}(Q_{\mathbf{r},\mathbf{t}}(q;n))\geq 0\ \forall b\gg 1. \tag{2.14}$$

The discussion of Section 2.5.1 also shows how to derive similar results for q-hypergeometric sequences of the form $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\leq 0}$.

3. The cyclotomic valuation of basic hypergeometric terms. In this section, we introduce some generalizations of Dwork maps and Landau functions. They provide suitable tools to respectively compute the ϕ_b-valuation of the q-Pochhammer symbol $(q^r;q^s)_n$ and of q-hypergeometric terms. Our approach takes its source in the works of Dwork [15], Katz [18], and Christol [11]. Precise formulas and properties for the p-adic valuation of Pochhammer symbols $(r/s)_n$ were given by Delaygue, Rivoal, and Roques [14] in order to prove the integrality of coefficients of some mirror maps. In this section, we generalize those formulas, yielding finer results in analogy with (1.6). We also show that our results extend naturally to negative n, and we derive new formulas that could be used to simplify the proofs in [14, Chapter 5] considerably.

3.1. A generalization of Dwork maps. We first extend the definition of the Dwork map D_p, replacing the prime number p by an arbitrary positive integer b.

For every rational number α, we let $d(\alpha)$ denote the exact positive denominator of α, that is $d(\alpha):=\min\{d\in\mathbb{N}:\alpha=a/d,\ a\in\mathbb{Z}\}$. Hence $d(\alpha)=1$ if and only if α is an integer. We also let $n(\alpha)$ denote the numerator of α, that is, the unique integer such that $\alpha=n(\alpha)/d(\alpha)$. For every positive integer b, we consider the multiplicative set $S_b:=\{k\in\mathbb{Z}:\gcd(k,b)=1\}$. We let $S_b^{-1}\mathbb{Z}\subset\mathbb{Q}$ denote the localization of $\mathbb{Z}$ by S_b, that is, the ring formed by the rational numbers α such that $d(\alpha)$ belongs to S_b.

Proposition-Definition 3.1. *Let b be a positive integer and α be in $S_b^{-1}\mathbb{Z}$. There is a unique element $D_b(\alpha)$ of $S_b^{-1}\mathbb{Z}$ such that*

$$bD_b(\alpha)-\alpha\in\{0,\dots,b-1\}. \tag{3.1}$$

Furthermore,

$$D_b(\alpha)=a\alpha+\left\lfloor\frac{\alpha-1}{b}-a\alpha\right\rfloor+1 \tag{3.2}$$

for every integer a satisfying $ab\equiv 1 \bmod d(\alpha)$.

REMARK 3.2. Note that the map D_b is only defined from $S_b^{-1}\mathbb{Z}$ into itself. When $b = 1$, $S_b^{-1}\mathbb{Z} = \mathbb{Q}$ and D_1 is just the identity map of $\mathbb{Q}$. In fact, not only $D_b(\alpha) \in S_b^{-1}\mathbb{Z}$, but, more precisely, equation (3.2) shows that $D_b(\alpha) \in \frac{1}{d(\alpha)}\mathbb{Z}$.

Proof of Proposition-Definition 3.1. Let us first assume for contradiction that $D_b(\alpha)$ is not unique, and let $\theta_1 > \theta_2$ be two distinct elements of $S_b^{-1}\mathbb{Z}$ satisfying (3.1). It would yield $b \geq 2$ and $b(\theta_1-\theta_2) \in \{1, \dots, b-1\}$. Therefore we would have $\theta_1 - \theta_2 \notin S_b^{-1}\mathbb{Z}$, contrary to $S_b^{-1}\mathbb{Z}$ being a ring. Hence $D_b(\alpha)$ is unique.

Now we prove the existence of $D_b(\alpha)$ while establishing (3.2). Since, by assumption, α belongs to $S_b^{-1}\mathbb{Z}$, we have $\gcd(d(\alpha), b) = 1$, and integers a such that $ab \equiv 1 \bmod d(\alpha)$ do exist. Let a be such an integer and set

$$\theta := a\alpha + \left\lfloor \frac{\alpha - 1}{b} - a\alpha \right\rfloor + 1.$$

Observe that $\theta \in S_b^{-1}\mathbb{Z}$. Since $ba \equiv 1 \bmod d(\alpha)$, $ba\alpha - \alpha$ is an integer and $b\theta - \alpha$ belongs to $\mathbb{Z}$. Furthermore,

$$\frac{\alpha - 1}{b} < \theta \leq \frac{\alpha - 1}{b} + 1,$$

which yields

$$-1 < b\theta - \alpha \leq b - 1.$$

Hence $D_b(\alpha) = \theta$, as expected. ∎

Following Christol [11], we introduce some notation which allows us to simplify the expression of $D_b(\alpha)$ when b is large enough. For every real number x, we let $\{x\}$ denote its fractional part and we set

$$\langle x \rangle := \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\langle x \rangle = 1 - \{1 - x\}$. For every rational number α, we also define

$$\mathfrak{n}_\alpha := \begin{cases} n(\alpha) & \text{if } \alpha \geq 0, \\ |n(\alpha)| + 1 & \text{otherwise.} \end{cases}$$

PROPOSITION 3.3. *Let b be a positive integer and α be in $S_b^{-1}\mathbb{Z}$. Let a be an integer satisfying $ab \equiv 1 \bmod d(\alpha)$. Then*

$$D_b(\alpha) = \langle a\alpha \rangle - \left\lfloor \langle a\alpha \rangle - \frac{\alpha}{b} \right\rfloor.$$

Furthermore, if $b \geq \mathfrak{n}_\alpha$, then

$$D_b(\alpha) = \begin{cases} \langle a\alpha \rangle & \textit{if } \alpha \notin \mathbb{Z}_{\leq 0}, \\ 0 & \textit{otherwise.} \end{cases}$$

It follows that, for a fixed rational number α, and for all integers $b \geq \mathfrak{n}_\alpha$ coprime to $d(\alpha)$, $D_b(\alpha)$ only depends on the residue class of b modulo $d(\alpha)$.

REMARK 3.4. When b is prime and $\alpha \notin \mathbb{Z}_{\leq 0}$, Lemma 23 in [14] shows that $D_b(\alpha) = \langle a\alpha\rangle$ for $b \geq d(\alpha)(|\lfloor 1-\alpha\rfloor| + \langle\alpha\rangle)$. The condition $b \geq \mathfrak{n}_\alpha$ slightly improves on this bound. When $\alpha > 0$, it makes no difference since $\mathfrak{n}_\alpha = n(\alpha)$ which can be written as $d(\alpha)\alpha = d(\alpha)(-\lfloor 1-\alpha\rfloor + \langle\alpha\rangle)$ with $\lfloor 1-\alpha\rfloor \leq 0$. But when $\alpha < 0$, we have $\mathfrak{n}_\alpha = |n(\alpha)| + 1$, which may improve on the previous bound. For example, even for $\alpha = -1/2$, one finds that $d(\alpha)(|\lfloor 1-\alpha\rfloor| + \langle\alpha\rangle) = 3$ while $\mathfrak{n}_\alpha = 2$.

Proof of Proposition 3.3. Since, by assumption, a does not divide $d(\alpha)$, there is an integer k such that $\langle a\alpha\rangle = k/d(\alpha)$ and $k \equiv an(\alpha) \bmod d(\alpha)$. Hence $bk \equiv n(\alpha) \bmod d(\alpha)$ and $b\langle a\alpha\rangle - \alpha$ is an integer. It follows that

$$b\left(\langle a\alpha\rangle - \left\lfloor \langle a\alpha\rangle - \frac{\alpha}{b}\right\rfloor\right) - \alpha \in \mathbb{Z}.$$

Furthermore,

$$\langle a\alpha\rangle - \frac{\alpha}{b} - 1 < \left\lfloor \langle a\alpha\rangle - \frac{\alpha}{b}\right\rfloor \leq \langle a\alpha\rangle - \frac{\alpha}{b},$$

so that

$$0 \leq b\left(\langle a\alpha\rangle - \left\lfloor \langle a\alpha\rangle - \frac{\alpha}{b}\right\rfloor\right) - \alpha < b.$$

This proves the expected formula for $D_b(\alpha)$ by uniqueness.

Now, let us assume that $b \geq \mathfrak{n}_\alpha$. Then $|\alpha/b| \leq 1/d(\alpha)$ and (even if α is an integer)

$$\frac{1}{d(\alpha)} \leq \langle a\alpha\rangle \leq 1.$$

If α is positive, then it follows that

$$\lfloor \langle a\alpha\rangle - \frac{\alpha}{b}\rfloor = 0, \tag{3.3}$$

that is, $D_b(\alpha) = \langle a\alpha\rangle$. If $\alpha = 0$, then $D_b(\alpha) = 0$. If α is negative, then $\mathfrak{n}_\alpha = |n(\alpha)| + 1$ and we obtain $|\alpha/b| < 1/d(\alpha)$. Hence, either α is an integer and

$$D_b(\alpha) = 1 - \left\lfloor 1 - \frac{\alpha}{b}\right\rfloor = 0,$$

or

$$\frac{1}{d(\alpha)} \leq \langle a\alpha\rangle \leq \frac{d(\alpha)-1}{d(\alpha)}$$

and $D_b(\alpha) = \langle a\alpha\rangle$. In all cases, we obtain the expected result. ∎

We end this section with a simple rule about composition of Dwork maps.

PROPOSITION 3.5. *Let b and c be positive integers, and let α be in $S_{bc}^{-1}\mathbb{Z}$. Then*

$$D_b(D_c(\alpha)) = D_{bc}(\alpha).$$

In particular, $D_b^n = D_{b^n}$, and if $b^n \geq \mathfrak{n}_\alpha$ is congruent to 1 modulo $d(\alpha)$ and $\alpha \notin \mathbb{Z}_{\leq 0}$, then $D_b^n(\alpha) = \langle\alpha\rangle$.

Proof. We have

$$bcD_b(D_c(\alpha)) - \alpha = c\big(bD_b(D_c(\alpha)) - D_c(\alpha)\big) + cD_c(\alpha) - \alpha,$$

which belongs to $\{0, \ldots, bc-1\}$. Hence $D_b(D_c(\alpha)) = D_{bc}(\alpha)$ by uniqueness. By induction, we get $D_b^n = D_{b^n}$. By Proposition 3.3, if $\alpha \notin \mathbb{Z}_{\leq 0}$ and $b^n \geq \mathfrak{n}_\alpha$ is congruent to 1 modulo $d(\alpha)$, then $D_b^n(\alpha) = D_{b^n}(\alpha) = \langle\alpha\rangle$. Indeed, since $b^n \equiv 1 \bmod d(\alpha)$, we can choose $a = 1$. ∎

3.2. The cyclotomic valuation of q-Pochhammer symbols. In this section, we rephrase Theorem 1.1 as Proposition 3.8 and then we prove the latter.

DEFINITION 3.6. Let r, s, and b be integers with $s \neq 0$ and $b \geq 1$. Set $\alpha := r/s$, $c := \gcd(r, s, b)$, $b' := b/c$, and $s' := s/c$. If $\gcd(s', b') = 1$, then $D_{b'}(\alpha)$ is well-defined and we set

$$\gamma := D_{b'}(\alpha) + \frac{\lfloor 1-\alpha \rfloor}{b'} \cdot \tag{3.4}$$

We define the (upper semicontinuous) step function $\delta_b(r, s, \cdot) : \mathbb{R} \to \mathbb{R}$ by

$$\delta_b(r, s, x) := \begin{cases} \lfloor cx - \gamma \rfloor + 1 & \text{if } \gcd(s', b') = 1, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.7. *The real number γ defined in* (3.4) *belongs to* $(0, 1]$.

Proof. By definition, $b'D_{b'}(\alpha) - \alpha$ belongs to $\{0, \ldots, b'-1\}$ and $\alpha = \langle\alpha\rangle - \lfloor 1-\alpha \rfloor$, where $\langle\alpha\rangle \in (0, 1]$. Thus,

$$0 < \frac{\langle\alpha\rangle}{b'} \leq D_{b'}(\alpha) + \frac{\lfloor 1-\alpha \rfloor}{b'} \leq \frac{b'-1+\langle\alpha\rangle}{b'} \leq 1,$$

as expected. ∎

PROPOSITION 3.8. *Let r, s, and b be integers such that $s \neq 0$ and $b \geq 1$. Let n be an integer such that $(q^r; q^s)_n$ is well-defined and non-zero. Then*

$$v_{\phi_b}((q^r; q^s)_n) = \delta_b(r, s, n/b).$$

It follows that if b divides both r and s, then $c = b$, $b' = 1$ and $\delta_b(r, s, n/b) = n$, as expected since $\phi_b(q)$ divides each factor $1 - q^{r+is}$. In particular, this is the case when $b = 1$.

In order to prove Proposition 3.8 for negative n, we need the following lemma. It is also used in the proof of our criterion for the q-integrality of q-hypergeometric sequences.

Lemma 3.9. *Let r, s, and n be integers with $s \neq 0$, and let b be a positive integer. Then*

$$\delta_b(r, s, -n/b) = -\delta_b(r-s, -s, n/b). \tag{3.5}$$

Proof. We set $c := \gcd(r, s, b)$ and write $b = cb'$ and $s = cs'$. Both sides of equation (3.5) are 0 when $\gcd(s', b') \neq 1$, so we can assume that s' and b' are coprime. Set $\alpha := r/s$ so that $1 - \alpha = (r-s)/(-s)$. We have

$$\delta_b(r-s, -s, x) = \left\lfloor cx - D_{b'}(1-\alpha) - \frac{\lfloor \alpha \rfloor}{b'} \right\rfloor + 1.$$

Since $b'D_{b'}(\alpha) - \alpha$ belongs to $\{0, \ldots, b'-1\}$, we have

$$b'(1 - D_{b'}(\alpha)) - (1-\alpha) = b' - 1 - (b'D_{b'}(\alpha) - \alpha) \in \{0, \ldots, b'-1\},$$

so that $D_{b'}(1-\alpha) = 1 - D_{b'}(\alpha)$. It follows that

$$\delta_b(r-s, -s, n/b) = \left\lfloor \frac{n}{b'} + D_{b'}(\alpha) - \frac{\lfloor \alpha \rfloor}{b'} \right\rfloor. \tag{3.6}$$

If $x \in \mathbb{R}$, we have $x \in \mathbb{Z}$ or $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$, which also yields $\alpha \in \mathbb{Z}$ or $\lfloor 1 - \alpha \rfloor = -\lfloor \alpha \rfloor$.

Let us first consider the case where $\alpha \notin \mathbb{Z}$. Then $\lfloor \alpha \rfloor = -\lfloor 1 - \alpha \rfloor$ and the right hand-side of (3.6) becomes

$$\left\lfloor \frac{n}{b'} + D_{b'}(\alpha) + \frac{\lfloor 1-\alpha \rfloor}{b'} \right\rfloor. \tag{3.7}$$

We have $n + b'D_{b'}(\alpha) - \alpha \in \mathbb{Z}$, but $\alpha \notin \mathbb{Z}$. Hence $n + b'D_{b'}(\alpha) + \lfloor 1 - \alpha \rfloor$ is not an integer and (3.7) is equal to

$$-\left\lfloor -\frac{n}{b'} - D_{b'}(\alpha) - \frac{\lfloor 1-\alpha \rfloor}{b'} \right\rfloor - 1 = -\delta_b(r, s, -n/b),$$

as expected.

It remains to consider the case where $\alpha \in \mathbb{Z}$. Set $k := -\delta_b(r, s, -n/b) \in \mathbb{Z}$. We have

$$\left\lfloor -\frac{n}{b'} - D_{b'}(\alpha) - \frac{\lfloor 1-\alpha \rfloor}{b'} \right\rfloor = -k - 1,$$

which yields the equivalences

$$\begin{aligned} -k-1 \le -\frac{n}{b'} - D_{b'}(\alpha) - \frac{\lfloor 1-\alpha \rfloor}{b'} < -k \\ \iff k < \frac{n}{b'} + D_{b'}(\alpha) + \frac{1-\alpha}{b'} \le k+1 \\ \iff k - \frac{1}{b'} < \frac{n}{b'} + D_{b'}(\alpha) - \frac{\alpha}{b'} \le k+1-\frac{1}{b'} \cdot \end{aligned}$$

Even if $b' = 1$, we obtain

$$\left\lfloor \frac{n}{b'} + D_{b'}(\alpha) - \frac{\alpha}{b'} \right\rfloor = k.$$

Combined with (3.6), this yields (3.5) and ends the proof of the lemma. ∎

Proof of Proposition 3.8. Set $r' = r/c$. We first consider the case $n \ge 0$. We assume that $(q^r; q^s)_n$ is non-zero, that is, $\alpha \notin \mathbb{Z}_{\le 0}$ or $n \le -\alpha$.

We observe that $b \mid (r + is)$ if and only if $b' \mid (r' + is')$. Since we have $\gcd(r', s', b') = 1$, if b' and s' are not coprime then $b \nmid (r+is)$ and $v_{\phi_b}((q^r; q^s)_n) = 0$.

We now assume that b' and s' are coprime. We need to find, among the powers of q in the product defining $(q^r; q^s)_n$, which are multiples of b. We have the following equivalences:

$$\begin{aligned} r' + is' \equiv 0 \bmod b' &\iff i \equiv -\alpha \bmod b' S_{b'}^{-1}\mathbb{Z} \\ &\iff i \equiv b' D_{b'}(\alpha) - \alpha \bmod b' \\ &\iff \exists k \in \mathbb{N}\ i = b' D_{b'}(\alpha) - \alpha + kb', \end{aligned}$$

because $i \ge 0$ and $b' D_{b'}(\alpha) - \alpha$ belongs to $\{0, \dots, b'-1\}$. We aim to count how many such integers i belong to $\{0, \dots, n-1\}$. Writing $n - 1 = v + mb'$, with $0 \le v \le b' - 1$, and setting $\eta := b' D_{b'}(\alpha) - \alpha$, we find that all the m integers $\eta, \eta + b', \dots, \eta + (m-1)b'$ can serve as i. There is one more such integer if and only if $v \ge b' D_{b'}(\alpha) - \alpha$. Furthermore,

$$v \ge b' D_{b'}(\alpha) - \alpha \iff v + 1 \ge b' D_{b'}(\alpha) + \lfloor 1 - \alpha \rfloor \tag{3.8}$$

$$\iff \frac{v+1}{b'} \ge D_{b'}(\alpha) + \frac{\lfloor 1-\alpha \rfloor}{b'} \cdot \tag{3.9}$$

Equivalence (3.8) follows from the implication

$$\begin{aligned} v + 1 \ge b' D_{b'}(\alpha) + \lfloor 1 - \alpha \rfloor &\implies v + 1 - \langle \alpha \rangle \ge b' D_{b'}(\alpha) - \alpha \\ &\implies v \ge b' D_{b'}(\alpha) - \alpha, \end{aligned}$$

because $1 - \langle \alpha \rangle$ belongs to $[0, 1)$. By Lemma 3.7, since both sides of (3.9) belong to $(0,1]$, we obtain

$$v_{\phi_b}((q^r;q^s)_n) = m + \left\lfloor \frac{v+1}{b'} - D_{b'}(\alpha) - \frac{\lfloor 1-\alpha \rfloor}{b'} \right\rfloor + 1$$
$$= \left\lfloor \frac{n}{b'} - D_{b'}(\alpha) - \frac{\lfloor 1-\alpha \rfloor}{b'} \right\rfloor + 1,$$

as expected.

We now assume that $n < 0$ and that $(q^r;q^s)_n$ is well-defined, that is, $\alpha \notin \mathbb{Z}_{>0}$ or $n > -\alpha$. We have

$$(q^r;q^s)_n = \frac{1}{(q^{r-s};q^{-s})_{-n}} \cdot$$

Using the non-negative case, we get $v_{\phi_b}((q^r;q^s)_n) = -\delta_b(r-s,-s,-n/b)$. By Lemma 3.9, the latter is equal to $\delta_b(r,s,n/b)$. This ends the proof. ∎

3.3. Extension to q-hypergeometric terms. Let

$$\mathbf{r} = ((r_1,s_1),\dots,(r_v,s_v)) \quad \text{and} \quad \mathbf{t} = ((t_1,u_1),\dots,(t_w,u_w))$$

be two vectors with integer coordinates and such that $s_1,\dots,s_v,u_1,\dots,u_w$ are non-zero. Set $\alpha_i := r_i/s_i$ and $\beta_j := t_j/u_j$. For every non-negative n, the ratio

$$Q_{\mathbf{r},\mathbf{t}}(q;n) = \frac{(q^{r_1};q^{s_1})_n \cdots (q^{r_v};q^{s_v})_n}{(q^{t_1};q^{u_1})_n \cdots (q^{t_w};q^{u_w})_n}$$

is well-defined if for each $j \in \{1,\dots,w\}$, we have either $\beta_j \notin \mathbb{Z}_{\le 0}$ or $n \le -\beta_j$. According to (1.8), this q-hypergeometric term admits the following extension to negative n:

$$Q_{\mathbf{r},\mathbf{t}}(q;n) = \frac{(q^{t_1-u_1};q^{-u_1})_{-n} \cdots (q^{t_w-u_w};q^{-u_w})_{-n}}{(q^{r_1-s_1};q^{-s_1})_{-n} \cdots (q^{r_v-s_v};q^{-s_v})_{-n}} \cdot$$

The latter is well-defined if for each $i \in \{1,\dots,v\}$, we have either $\alpha_i \notin \mathbb{Z}_{>0}$ or $n > -\alpha_i$.

If $R(q)$ and $S(q)$ are non-zero elements in $\mathbb{Z}[q^{-1},q]$, we write $R(q) \sim S(q)$ when $R(q)/S(q)$ is a unit of $\mathbb{Z}[q^{-1},q]$, that is, when it is of the form ϵq^m with $\epsilon \in \{-1,1\}$ and $m \in \mathbb{Z}$.

We now introduce some step functions that generalize the Landau functions mentioned in the introduction.

Definition 3.10. Keeping the notation of Section 3.2, for every integer b we define the (upper semicontinuous) step function $\Delta_b^{\mathbf{r},\mathbf{t}} : \mathbb{R} \to \mathbb{R}$ by

$$\Delta_b^{\mathbf{r},\mathbf{t}}(x) := \sum_{i=1}^{v} \delta_b(r_i,s_i,x) - \sum_{j=1}^{w} \delta_b(t_j,u_j,x).$$

As a direct consequence of Proposition 3.8, we deduce the following result.

Corollary 3.11. *Let $n \in \mathbb{Z}$ be such that $Q_{\mathbf{r},\mathbf{t}}(q;n)$ is well-defined and non-zero. Then*

$$v_{\phi_b}(Q_{\mathbf{r},\mathbf{t}}(q;n)) = \Delta_b^{\mathbf{r},\mathbf{t}}(n/b),$$

that is,

$$Q_{\mathbf{r},\mathbf{t}}(q;n) \sim \prod_{b=1}^{\infty} \phi_b(q)^{\Delta_b^{\mathbf{r},\mathbf{t}}(n/b)}. \tag{3.10}$$

Remark 3.12. Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_v)$ and $\boldsymbol{\beta} := (\beta_1, \dots, \beta_w)$ be vectors of rational numbers. Let $d := d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be the least common multiple of the denominators of the rational numbers α_i and β_j. Let n be an integer such that

$$\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;n) := \frac{(q^{\alpha_1};q)_n \cdots (q^{\alpha_v};q)_n}{(q^{\beta_1};q)_n \cdots (q^{\beta_w};q)_n} \tag{3.11}$$

is well-defined and non-zero. Then $\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;n)$ belongs to $\mathbb{Q}(q^{1/d})$. By Remark 2.1, Corollary 3.11 implies that

$$\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;n) \sim \prod_{b=1}^{\infty} \phi_b(q^{1/d})^{\Delta_b^{\mathbf{r},\mathbf{t}}(n/b)}, \tag{3.12}$$

where $\sim$ has to be understood in $\mathbb{Z}[q^{-1/d}, q^{1/d}]$, and where

$$\mathbf{r} = ((d\alpha_1, d), \dots, (d\alpha_v, d)) \quad \text{and} \quad \mathbf{t} = ((d\beta_1, d), \dots, (d\beta_w, d)).$$

4. First criteria for q-integrality of basic hypergeometric sequences. In this section, we provide a criterion for the q-integrality of the q-hypergeometric sequences in terms of the Landau functions $\Delta_b^{\mathbf{r},\mathbf{t}}$, as well as related results.

4.1. A first criterion of q-integrality. Our first result reads as follows.

Proposition 4.1. *Keeping the notation of the previous sections, assume that $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is a well-defined sequence. Then the following two assertions are equivalent:*

(i) *There exists $C(q) \in \mathbb{Z}[q] \setminus \{0\}$ such that $C(q)^n Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1}, q]$ for every $n \geq 0$.*
(ii) *For all but finitely many positive integers b, $\Delta_b^{\mathbf{r},\mathbf{t}}$ is non-negative on $\mathbb{R}_{\geq 0}$.*

According to (2.13), we deduce from Proposition 4.1 the following result.

Corollary 4.2. *Assume that $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is a well-defined sequence. Let $\mathcal{N}_1 := \{i \in \{1, \dots, v\} : s_i < 0\}$, $\mathcal{N}_2 := \{j \in \{1, \dots, w\} : u_j < 0\}$, and $s = \sum_{i\in\mathcal{N}_1} s_i - \sum_{j\in\mathcal{N}_2} u_j$. Assume that $s \geq 0$. Then the following two assertions are equivalent.*

(i) *The sequence $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is q-integral.*
(ii) *For all but finitely many positive integers b, $\Delta_b^{\mathbf{r},\mathbf{t}}$ is non-negative on $\mathbb{R}_{\geq 0}$.*

Throughout this section, we fix $\mathbf{r}$ and $\mathbf{t}$, and we write Δ_b as shorthand for $\Delta_b^{\mathbf{r},\mathbf{t}}$. Before proving Proposition 4.1, we need to establish the following lemma about the jumps of Landau step functions.

Lemma 4.3. *For any integers k and $b \geq 1$, and every real number x, we have*

$$\Delta_b(x+k) = \Delta_b(x) + k\Delta_b(1).$$

Furthermore, if b is large enough, then the distance between any two distinct jumps of Δ_b is greater than or equal to $1/b$.

Remark 4.4. By Lemma 4.3, Δ_b is non-negative on $\mathbb{R}_{\geq 0}$ if and only if it is non-negative on $[0,1]$. In addition, if b is coprime to $d_{\mathbf{r},\mathbf{t}}$, then $\Delta_b(1) = v - w$ and Proposition 4.1(ii) implies that $v \geq w$.

Proof of Lemma 4.3. Let us first give a useful expression for Δ_b. For all i and j, we recall that $\alpha_i = r_i/s_i$ and $\beta_j = t_j/u_j$. We also set $c_i := \gcd(r_i, s_i, b)$, $d_j := \gcd(t_j, u_j, b)$, and

$$\begin{aligned} V_b &:= \{1 \leq i \leq v : \gcd(s_i, b) = c_i\}, \\ W_b &:= \{1 \leq j \leq w : \gcd(u_j, b) = d_j\}. \end{aligned} \tag{4.1}$$

We observe that $i \in V_b$ if and only if $\delta_b(r_i, s_i, \cdot)$ is not the zero function, while $j \in W_b$ if and only if $\delta_b(t_j, u_j, \cdot)$ is not the zero function. It follows that

$$\begin{aligned} \Delta_b(x) = &\sum_{i \in V_b} \left\lfloor c_i x - D_{b/c_i}(\alpha_i) - \frac{\lfloor 1-\alpha_i \rfloor}{b/c_i} \right\rfloor \\ &- \sum_{j \in W_b} \left\lfloor d_j x - D_{b/d_j}(\beta_j) - \frac{\lfloor 1-\beta_j \rfloor}{b/d_j} \right\rfloor + \#V_b - \#W_b. \end{aligned} \tag{4.2}$$

Since b/c_i is coprime to $d(\alpha_i)$ and b/d_j is coprime to $d(\beta_j)$, we infer from Lemma 3.7 that

$$D_{b/c_i}(\alpha_i) + \frac{\lfloor 1-\alpha_i \rfloor}{b/c_i} \in (0,1] \quad \text{and} \quad D_{b/d_j}(\beta_i) + \frac{\lfloor 1-\beta_j \rfloor}{b/d_j} \in (0,1].$$

By (4.2), we first deduce that $\Delta_b(1) = \sum_{i \in V_b} c_i - \sum_{j \in W_b} d_j$, and so

$$\begin{aligned} \Delta_b(x+k) &= \Delta_b(x) + \sum_{i \in V_b} c_i k - \sum_{j \in W_b} d_j k \\ &= \Delta_b(x) + k\Delta_b(1) \end{aligned}$$

for every integer k. This proves the first part of the lemma.

By (4.2), the jumps of the step function Δ_b have abscissas of the form

$$\gamma(r,s,k) := \frac{D_{b/c}(\alpha) + k}{c} + \frac{\lfloor 1-\alpha \rfloor}{b}, \tag{4.3}$$

where (r,s) belongs to $\mathbf{r}$ or $\mathbf{t}$, $\alpha = r/s$, $c = \gcd(r,s,b)$ and $k \in \mathbb{Z}$. Let $\gamma_1 := \gamma(r_1, s_1, k_1)$ and $\gamma_2 := \gamma(r_2, s_2, k_2)$ be two distinct abscissas of jumps as in (4.3). For $i = 1$ or 2, set $\alpha_i := r_i/s_i$, $c_i := \gcd(r_i, s_i, b)$ and $b_i := b/c_i$. If

$$\frac{D_{b_1}(\alpha_1)+k_1}{c_1} = \frac{D_{b_2}(\alpha_2)+k_2}{c_2},$$

then $\lfloor 1-\alpha_1\rfloor \neq \lfloor 1-\alpha_2\rfloor$ and

$$|\gamma_1-\gamma_2| = \frac{|\lfloor 1-\alpha_1\rfloor - \lfloor 1-\alpha_2\rfloor|}{b} \geq \frac{1}{b},$$

as expected. Otherwise, we get

$$\left|\frac{D_{b_1}(\alpha_1)+k_1}{c_1} - \frac{D_{b_2}(\alpha_2)+k_2}{c_2}\right| \geq \frac{1}{d_{\mathbf{r},\mathbf{t}}}\,.$$

Indeed, we infer from Remark 3.2 that $D_{b_i}(\alpha_i) \in \frac{c_i}{s_i}\mathbb{Z}$, which shows that

$$\frac{D_{b_i}(\alpha_i)+k_i}{c_i} \in \frac{1}{s_i}\mathbb{Z}.$$

Hence, for $b \geq 2d_{\mathbf{r},\mathbf{t}} \cdot \max\{|\lfloor 1-\alpha\rfloor - \lfloor 1-\beta\rfloor| + 1 : \alpha \text{ and } \beta \text{ in } \boldsymbol{\alpha} \text{ or } \boldsymbol{\beta}\}$, we have

$$|\gamma_1-\gamma_2| > \frac{1}{2d_{\mathbf{r},\mathbf{t}}} \geq \frac{1}{b},$$

as expected. This ends the proof. ∎

Proof of Proposition 4.1. We first infer from (2.14) and (3.10) that assertion (ii) implies (i). Now, we assume that (i) holds and we prove (ii). By (2.14) and (3.10), there exists a positive integer m such that, for every non-negative integer n and every integer $b \geq m$, we have $\Delta_b(n/b) \geq 0$. By Lemma 4.3, we can assume that m is such that, for $b \geq m$, the distance between any two distinct jumps of Δ_b is greater than or equal to $1/b$. It follows that Δ_b is non-negative on $\mathbb{R}_{\geq 0}$ for all $b \geq m$, as desired. ∎

4.2. Related criteria for negative arguments. It is easy to deduce from Proposition 4.1 a criterion for the q-integrality of the sequence $(\mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;-n))_{n\geq 0}$. Indeed, for every integer n, we have $\mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n) = \mathcal{Q}_{\mathbf{t}',\mathbf{r}'}(q;-n)$ (assuming that both terms are well-defined), where $\mathbf{r}'$ and $\mathbf{t}'$ are respectively obtained from $\mathbf{r}$ and $\mathbf{t}$ by replacing each pair (r,s) in $\mathbf{r}$ or $\mathbf{t}$ by $(r-s,-s)$. By Lemma 3.9, for every positive integer b, we have

$$\Delta_b^{\mathbf{r},\mathbf{t}}(n/b) = \Delta_b^{\mathbf{t}',\mathbf{r}'}(-n/b).$$

Combining Lemma 4.3 and Proposition 4.1, we find that the following two assertions are equivalent.

(i) There exists $C(q) \in \mathbb{Z}[q] \setminus \{0\}$ such that $C(q)^n \mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$ for every $n \in \mathbb{Z}_{\leq 0}$.

(ii) For all but finitely many positive integers b, $\Delta_b^{\mathbf{r},\mathbf{t}}$ is non-negative on $\mathbb{R}_{\leq 0}$.

A natural question is then whether it is possible to find a non-zero rational fraction $C(q)$ in $\mathbb{Q}(q)$ such that $C(q)^n Q_{\mathbf{r},\mathbf{t}}(q;n)$ is a polynomial for positive and negative n simultaneously. The main problem is that the numerator of $C(q)$ will bring new denominators for negative n and *vice versa*. It turns out that this problem can be overcome only in the special case where $Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$ for all integers n.

Proposition 4.5. *Assume that $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\in\mathbb{Z}}$ is a well-defined family. Then the following three assertions are equivalent:*

(i) *There exists $C(q) \in \mathbb{Q}[q] \setminus \{0\}$ such that $C(q)^n Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$ for every $n \in \mathbb{Z}$.*
(ii) *For every $n \in \mathbb{Z}$, $Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$.*
(iii) *For every $n \in \mathbb{N}$, $Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$, and for all but finitely many positive integers b, $\Delta_b^{\mathbf{r},\mathbf{t}}$ is 1-periodic.*

Proof. Let us first prove that (i) implies (iii). If we assume (i), then, by the above criteria, for every large enough positive integer b, Δ_b is non-negative on $\mathbb{R}$. By Lemma 4.3, we deduce that $\Delta_b(1) = 0$ and Δ_b is 1-periodic. Even for small positive integers b, we have

$$\Delta_b(1) = \sum_{i\in V_b} c_i - \sum_{j\in W_b} d_j,$$

where V_b, W_b, c_i and d_j are defined as in (4.1), and only depend on the congruence class of b modulo $d_{\mathbf{r},\mathbf{t}}$. Hence $\Delta_b(1) = \Delta_{b+ld_{\mathbf{r},\mathbf{t}}}(1)$; but $\Delta_{b+ld_{\mathbf{r},\mathbf{t}}}(1) = 0$ for l large enough, so $\Delta_b(1) = 0$ and Δ_b is 1-periodic for every positive integer b. In particular, if $\Delta_b(n/b) < 0$ for some positive integers n and b, then there exists a negative integer m such that $\Delta_b(m/b) < 0$. In this case, the ϕ_b-valuations of both $Q_{\mathbf{r},\mathbf{t}}(q;n)$ and $Q_{\mathbf{r},\mathbf{t}}(q;m)$ are negative, which contradicts (i). It follows that $Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$ for every $n \in \mathbb{N}$, and (iii) is proved.

Now, let us prove that (iii) implies (ii). If (iii) holds, then, reasoning as above, we find that Δ_b is 1-periodic for all positive integers b. For all positive integers n and b, we have $Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$, so that $\Delta_b(n/b) \geq 0$. By 1-periodicity, for all integers n and $b \geq 1$, we have $\Delta_b(n/b) \geq 0$, that is, $Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1},q]$, as expected.

Obviously, (ii) implies (i) by choosing $C(q) = 1$, which ends the proof of the proposition. ∎

4.3. Small digression on the step function $\Delta_b^{\mathbf{r},\mathbf{t}}$. In this section, we use Proposition 3.3 to simplify the expression of $\Delta_b(x)$ when b is large enough. To that end we introduce some additional notation. Keeping the notation introduced in (4.1), we let $\mathfrak{n}_\alpha$ be defined as in Proposition 3.3. We define $\mathfrak{a}_{\mathbf{r},\mathbf{t}}$ as the maximum of the numbers $\gcd(r_i,s_i)$ and $\gcd(t_j,u_j)$ for all

i and j. We set

$$\mathfrak{n}_{\mathbf{r},\mathbf{t}} := \max\{\mathfrak{n}_\alpha : \alpha \text{ in } \boldsymbol{\alpha} \text{ or } \boldsymbol{\beta}\} \quad \text{and} \quad \mathfrak{b}_{\mathbf{r},\mathbf{t}} := \mathfrak{a}_{\mathbf{r},\mathbf{t}} \cdot \mathfrak{n}_{\mathbf{r},\mathbf{t}}.$$

Let $b \geq \mathfrak{b}_{\mathbf{r},\mathbf{t}}$ be a fixed integer. For every $i \in V_b$ and $j \in W_b$, there exist positive integers e_i and f_j such that

$$be_i \equiv c_i \bmod s_i \quad \text{and} \quad bf_j \equiv d_j \bmod u_j.$$

Now, take for example $i \in V_b$. We have

$$\frac{b}{c_i} \geq \frac{\mathfrak{b}_{\mathbf{r},\mathbf{t}}}{c_i} \geq \frac{\mathfrak{a}_{\mathbf{r},\mathbf{t}}}{c_i}\mathfrak{n}_{\mathbf{r},\mathbf{t}} \geq \mathfrak{n}_{\alpha_i}.$$

So we can apply Proposition 3.3 to find that $D_{b/c_i}(\alpha_i) = \langle e_i\alpha_i\rangle$ if $\alpha_i \notin \mathbb{Z}_{\leq 0}$ and 0 otherwise. Let us consider a slight modification of the function $\langle\cdot\rangle$ defined for every $x \in \mathbb{R}$ by

$$\langle x\rangle^* := \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z}, \\ 1 & \text{if } x \in \mathbb{Z}_{>0}, \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in V_b$, if $c_i < s_i$, then e_i is invertible modulo s_i/c_i which is a denominator of α_i. It follows that $e_i\alpha_i \in \mathbb{Z}_{\leq 0}$ if and only if $\alpha_i \in \mathbb{Z}_{\leq 0}$. Hence, we deduce from (4.2) that, for all $b \geq \mathfrak{b}_{\mathbf{r},\mathbf{t}}$ and all $x \in \mathbb{R}$, $\Delta_b(x)$ is equal to

$$\sum_{i\in V_b}\left\lfloor c_ix - \langle e_i\alpha_i\rangle^* - \frac{\lfloor 1-\alpha_i\rfloor}{b/c_i}\right\rfloor - \sum_{j\in W_b}\left\lfloor d_jx - \langle f_j\beta_j\rangle^* - \frac{\lfloor 1-\beta_j\rfloor}{b/d_j}\right\rfloor + \#V_b - \#W_b. \tag{4.4}$$

Let $d_{\mathbf{r},\mathbf{t}}$ be the least common multiple of the integers $s_1,\dots,s_v,u_1,\dots,u_w$. If in addition b is coprime to $d_{\mathbf{r},\mathbf{t}}$, then all the numbers c_i and d_j are equal to 1. Let a in $\{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$ be such that $ab \equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$. Then, for all i and j, we can take $e_i = f_j = a$, so that

$$\Delta_b(x) = \sum_{i=1}^{v}\left\lfloor x - \langle a\alpha_i\rangle^* - \frac{\lfloor 1-\alpha_i\rfloor}{b}\right\rfloor - \sum_{j=1}^{w}\left\lfloor x - \langle a\beta_j\rangle^* - \frac{\lfloor 1-\beta_j\rfloor}{b}\right\rfloor + v - w.$$

Moreover, if all the numbers α_i and β_j belong to $(0,1]$, then

$$\Delta_b(x) = \sum_{i=1}^{v}\lfloor x - \langle a\alpha_i\rangle\rfloor - \sum_{j=1}^{w}\lfloor x - \langle a\beta_j\rangle\rfloor + v - w,$$

which only depends on the congruence class of b modulo $d_{\mathbf{r},\mathbf{t}}$.

5. Efficient criteria for q-integrality of basic hypergeometric sequences. To verify the second assertion in Proposition 4.1 and in Corollary

4.2, we need in principle to perform infinitely many tests, checking the non-negativity of the step function $\Delta_b^{\mathbf{r},\mathbf{t}}$ on $\mathbb{R}_{\geq 0}$ for all sufficiently large integers b. This is not entirely satisfactory and the aim of Theorem 1.3 is precisely to reduce the situation to a finite number of similar tests. In this section, we introduce the step functions $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$, $b \in \{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$. Then we prove Theorem 1.3.

5.1. A generalization of Christol step functions. Following Christol [11], we define a total order $\preceq$ on $\mathbb{R}$ as follows. For all real numbers x and y, we set

$$x \preceq y \iff (\langle x\rangle < \langle y\rangle \text{ or } (\langle x\rangle = \langle y\rangle \text{ and } x \geq y)).$$

We refer to it as *Christol order*. Let $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_v)$ and $\boldsymbol{\beta} := (\beta_1,\dots,\beta_w)$ be two vectors of rational numbers, and

$$d_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \operatorname{lcm}(d(\alpha_1),\dots,d(\alpha_v),d(\beta_1),\dots,d(\beta_w)).$$

For every integer $a \in \{1,\dots,d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, Christol defined a step function $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$ from $\mathbb{R}$ to $\mathbb{R}$ by

$$\begin{aligned}\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x) := {} & \#\{i \in \{1,\dots,v\} : a\alpha_i \preceq x\} \\ & - \#\{j \in \{1,\dots,w\} : a\beta_j \preceq x\}.\end{aligned} \tag{5.1}$$

We recall here our notation. Let v and w be positive integers, and for $i \in \{1,\dots,v\}$ and $j \in \{1,\dots,w\}$, let (r_i,s_i) and (t_j,u_j) be pairs of integers such that $s_iu_j \neq 0$ for all (i,j). Set $\alpha_i := r_i/s_i$, $\beta_j := t_j/u_j$, $\mathbf{r} := ((r_1,s_1),\dots,(r_v,s_v))$, $\mathbf{t} := ((t_1,u_1),\dots,(t_w,u_w))$, $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_v)$, $\boldsymbol{\beta} := (\beta_1,\dots,\beta_w)$, and $d_{\mathbf{r},\mathbf{t}} := \operatorname{lcm}(s_1,\dots,s_v,u_1,\dots,u_w)$.

For every $b \in \{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$, we define the step function $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ as follows. For all $i \in \{1,\dots,v\}$ and $j \in \{1,\dots,w\}$, we set $c_i := \gcd(r_i,s_i,b)$ and $d_j := \gcd(t_j,u_j,b)$. We consider, as in (4.1), the sets of indices

$$\begin{aligned}V_b &:= \{1 \leq i \leq v : \gcd(s_i,b) = c_i\},\\ W_b &:= \{1 \leq j \leq w : \gcd(u_j,b) = d_j\}.\end{aligned}$$

As already observed in Section 4.3, for every $i \in V_b$ and $j \in W_b$, there exist positive integers e_i and f_j such that

$$be_i \equiv c_i \bmod s_i \quad \text{and} \quad bf_j \equiv d_j \bmod u_j.$$

For all i,j, we choose such integers e_i and f_j. We stress that the definition of $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ (see Definition 5.1) does not depend on this choice. Let $\tilde{b}$ be the greatest divisor of b coprime to $d_{\mathbf{r},\mathbf{t}}$ and let a be the unique element of $\{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$ satisfying $a\tilde{b} \equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$.

Definition 5.1. For every integer b in $\{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$, we define the step function $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot):\mathbb{R}\to\mathbb{R}$ by

$$\Xi_{\mathbf{r},\mathbf{t}}(b,x) := \#\left\{(i,k)\in V_b\times\{0,\dots,c_i-1\} : \frac{\langle e_i\alpha_i\rangle+k}{c_i}-\lfloor 1-a\alpha_i\rfloor \preceq x\right\} - \#\left\{(j,\ell)\in W_b\times\{0,\dots,d_j-1\} : \frac{\langle f_j\beta_j\rangle+\ell}{d_j}-\lfloor 1-a\beta_j\rfloor \preceq x\right\}.$$

5.2. Comparison with the step functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$ and $\Delta_b^{\mathbf{r},\mathbf{t}}$. The functions $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ can be thought of as a generalization of $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$ to composite numbers b. Indeed, if we assume that b is coprime to $d_{\mathbf{r},\mathbf{t}}$ and that all the ratios $\alpha_i=r_i/s_i$ and $\beta_j=t_j/u_j$ belong to $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$, then $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)=\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,\cdot)$ where $ab\equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$.

Let us prove this claim. If b is coprime to $d_{\mathbf{r},\mathbf{t}}$, then $b=\tilde{b}$, all the numbers c_i and d_j are equal to 1, $V_b=\{1,\dots,v\}$, and $W_b=\{1,\dots,w\}$. Hence, for all i and j, we can choose $e_i=f_j=a$. Moreover, for all $(i,k)\in V_b\times\{0,\dots,c_i-1\}$, we have $k=0$. We obtain

$$\frac{\langle e_i\alpha_i\rangle+k}{c_i}-\lfloor 1-a\alpha_i\rfloor=\langle a\alpha_i\rangle-\lfloor 1-a\alpha_i\rfloor=a\alpha_i.$$

Similarly, for all $(j,\ell)\in W_b\times\{0,\dots,d_j-1\}$, we have

$$\frac{\langle f_j\beta_j\rangle+\ell}{d_j}-\lfloor 1-a\beta_j\rfloor=a\beta_j.$$

By (5.1), we get

$$\begin{aligned}\Xi_{\mathbf{r},\mathbf{t}}(b,x)&=\#\big\{(i,k)\in\{1,\dots,v\}\times\{0\}:a\alpha_i\preceq x\big\}\\&\qquad-\#\big\{(j,\ell)\in\{1,\dots,w\}\times\{0\}:a\beta_j\preceq x\big\}\\&=\#\big\{i\in\{1,\dots,v\}:a\alpha_i\preceq x\big\}-\#\big\{j\in\{1,\dots,w\}:a\beta_j\preceq x\big\}\\&=\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a,x).\end{aligned}$$

Let us now compare the step functions $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ and $\Delta_b^{\mathbf{r},\mathbf{t}}$. Using (4.2), we can give a new expression for $\Delta_b^{\mathbf{r},\mathbf{t}}$ (restricted on $[0,1]$) which is closer to the definition of $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$. Indeed, for every positive integer b and every real number x in $[0,1]$, we get

(5.2)
$$\Delta_b^{\mathbf{r},\mathbf{t}}(x)=\#\left\{(i,k)\in V_b\times\{0,\dots,c_i-1\}:\frac{D_{b/c_i}(\alpha_i)+k}{c_i}+\frac{\lfloor 1-\alpha_i\rfloor}{b}\leq x\right\} - \#\left\{(j,\ell)\in W_b\times\{0,\dots,d_j-1\}:\frac{D_{b/d_j}(\beta_i)+\ell}{d_j}+\frac{\lfloor 1-\beta_j\rfloor}{b}\leq x\right\}.$$

5.3. Ordering of jumps. The interest of the step functions $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ is that they keep track of all jumps configurations of the Landau functions $\Delta_\ell^{\mathbf{r},\mathbf{t}}$

for large ℓ congruent to b modulo $d_{\mathbf{r},\mathbf{t}}$. More precisely, we have the following result.

LEMMA 5.2. *For every $i \in \{1,2\}$, let r_i and s_i be integers with $s_i \neq 0$ and such that $\alpha_i := r_i/s_i \notin \mathbb{Z}_{\leq 0}$. Set $d := \mathrm{lcm}(s_1, s_2)$ and let b be an integer such that*

$$b > \max(|r_1|, |r_2|, d \cdot |\lfloor 1-\alpha_1 \rfloor - \lfloor 1-\alpha_2 \rfloor|).$$

Set $c_i := \gcd(r_i, s_i, b)$ and assume that there exists an integer e_i, $1 \leq e_i \leq d$, such that $be_i \equiv c_i \bmod s_i$. Let $k_i \in \{0,\dots,c_i-1\}$ and a be a positive integer. Set

$$\gamma_i := \frac{D_{b/c_i}(\alpha_i)+k_i}{c_i} + \frac{\lfloor 1-\alpha_i \rfloor}{b} \quad \text{and} \quad \varGamma_i := \frac{\langle e_i\alpha_i\rangle + k_i}{c_i} - \lfloor 1 - a\alpha_i \rfloor.$$

Then

$$\gamma_1 \leq \gamma_2 \iff \varGamma_1 \preceq \varGamma_2.$$

Furthermore, if $\varGamma_1 = \varGamma_2$, then $\alpha_1 = \alpha_2$.

REMARK 5.3. Contrary to what the notation of Lemma 5.2 may suggest, we stress that this lemma applies to compare the ordering of both the jumps with positive and negative amplitude of the step functions $\varXi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ and $\varDelta_b^{\mathbf{r},\mathbf{t}}$.

Even when $b \geq \mathfrak{b}_{\mathbf{r},\mathbf{t}}$, formula (4.4) shows that the Landau functions $\varDelta_b^{\mathbf{r},\mathbf{t}}$ depend in principle on b and not only on the congruence class of b modulo $d_{\mathbf{r},\mathbf{t}}$. In contrast, Lemma 5.2 shows that for sufficiently large b, the $\leq$-ordering of the jumps of $\varDelta_b^{\mathbf{r},\mathbf{t}}$ on $[0,1]$ is the same as the $\preceq$-ordering of the jumps of $\varXi_{\mathbf{r},\mathbf{t}}(\underline{b},\cdot)$ on $\mathbb{R}$, where $\underline{b}$ is the unique representative in $\{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$ of b modulo $d_{\mathbf{r},\mathbf{t}}$. In particular, this ordering only depends on the congruence class of b modulo $d_{\mathbf{r},\mathbf{t}}$.

Furthermore, Lemma 5.2 shows that if two jumps of $\varXi_{\mathbf{r},\mathbf{t}}(b,\cdot)$, respectively associated with the pairs (r_1,s_1) and (r_2,s_2), have the same abscissa, then we must have $r_1/s_1 = r_2/s_2$. However, these pairs can still be distinct. Indeed, taking for example the pairs $(r_1,s_1) = (1,4)$ and $(r_2,s_2) = (3,12)$, and $b = 9$, we find that $d = 12$ and $\tilde{b} = 1$, so that $a = 1$, $c_1 = 1$, $c_2 = 3$, $e_1 = 1$, and $e_2 = 3$. Hence taking $k_1 = k_2 = 0$ yields

$$\varGamma_1 = \langle 1/4 \rangle = \frac{1}{4} \quad \text{and} \quad \varGamma_2 = \frac{\langle 3/4 \rangle}{3} = \frac{1}{4}\cdot$$

Proof of Lemma 5.2. For $i \in \{1,2\}$, we set $b_i := b/c_i$ and

$$\theta_i := \frac{D_{b_i}(\alpha_i)+k_i}{c_i}\cdot$$

Since $b > |r_i|$ and c_i divides both r_i and s_i, we have $b_i > |n(\alpha_i)|$ and hence $b_i \geq \mathfrak{n}_{\alpha_i}$. By Proposition 3.3, we have $D_{b_i}(\alpha_i) = \langle e_i\alpha_i\rangle$ for $\alpha_i \notin \mathbb{Z}_{\leq 0}$, so that

$$\theta_i = \frac{\langle e_i\alpha_i\rangle + k_i}{c_i}\cdot \tag{5.3}$$

Note that $\theta_i \in \frac{1}{d}\mathbb{Z}$. Indeed, c_i divides $\gcd(r_i, s_i)$ so that $\alpha_i/c_i \in \frac{1}{s_i}\mathbb{Z}$, while d is a multiple of s_i. Now, we show that

$$\theta_1 = \theta_2 \implies \langle\alpha_1\rangle = \langle\alpha_2\rangle. \tag{5.4}$$

Setting $s_i' := s_i/c_i$ for $i \in \{1,2\}$, we see that $b_i e_i \equiv 1 \bmod s_i'$ and $\alpha_i \in \frac{1}{s_i'}\mathbb{Z}$. We obtain

$$\langle b_i\langle e_i\alpha_i\rangle\rangle = \langle\alpha_i\rangle.$$

Therefore,

$$\begin{aligned}\theta_1 = \theta_2 &\implies b\theta_1 = b\theta_2\\ &\implies b_1\langle e_1\alpha_1\rangle + b_1k_1 = b_2\langle e_2\alpha_2\rangle + b_2k_2\\ &\implies \langle b_1\langle e_1\alpha_1\rangle\rangle = \langle b_2\langle e_2\alpha_2\rangle\rangle\\ &\implies \langle\alpha_1\rangle = \langle\alpha_2\rangle,\end{aligned}$$

which proves (5.4).

Furthermore, since $\alpha = \langle\alpha\rangle - \lfloor 1-\alpha\rfloor$, we have

$$\lfloor 1 - a\alpha_i\rfloor = \lfloor 1 - a\langle\alpha_i\rangle + a\lfloor 1-\alpha_i\rfloor\rfloor = \lfloor 1 - a\langle\alpha_i\rangle\rfloor + a\lfloor 1-\alpha_i\rfloor.$$

If $\theta_1 = \theta_2$, then $\langle\alpha_1\rangle = \langle\alpha_2\rangle$ and

$$\lfloor 1 - a\alpha_1\rfloor \geq \lfloor 1 - a\alpha_2\rfloor \iff \lfloor 1-\alpha_1\rfloor \geq \lfloor 1-\alpha_2\rfloor, \tag{5.5}$$

for a is a positive integer. Since $\theta_i \in \frac{1}{d}\mathbb{Z}$ and $b > d\cdot\big|\lfloor 1-\alpha_1\rfloor - \lfloor 1-\alpha_2\rfloor\big|$, we obtain the following equivalences:

$$\begin{aligned}\gamma_1 \leq \gamma_2 &\iff \theta_1 + \frac{\lfloor 1-\alpha_1\rfloor}{b} \leq \theta_2 + \frac{\lfloor 1-\alpha_2\rfloor}{b}\\ &\iff \theta_1 - \theta_2 \leq \frac{\lfloor 1-\alpha_2\rfloor - \lfloor 1-\alpha_1\rfloor}{b}\\ &\iff \theta_1 < \theta_2 \text{ or } (\theta_1 = \theta_2 \text{ and } \lfloor 1-\alpha_1\rfloor \leq \lfloor 1-\alpha_2\rfloor)\\ &\iff \theta_1 < \theta_2 \text{ or } (\theta_1 = \theta_2 \text{ and } \lfloor 1-a\alpha_1\rfloor \leq \lfloor 1-a\alpha_2\rfloor)\\ &\iff \theta_1 - \lfloor 1-a\alpha_1\rfloor \preceq \theta_2 - \lfloor 1-a\alpha_2\rfloor\\ &\iff \mathit{\Gamma}_1 \preceq \mathit{\Gamma}_2.\end{aligned}$$

Indeed, we have $\theta_i \in (0,1]$, which implies that $\langle\theta_i - \lfloor 1-a\alpha_i\rfloor\rangle = \theta_i$, while (5.3) implies that $\theta_i - \lfloor 1-a\alpha_i\rfloor = \mathit{\Gamma}_i$. This proves the first part of the proposition.

Now, assume that $\mathit{\Gamma}_1 = \mathit{\Gamma}_2$ so that $\theta_1 - \lfloor 1-a\alpha_1\rfloor = \theta_2 - \lfloor 1-a\alpha_2\rfloor$. Since $\theta_i \in (0,1]$, it follows that $\theta_1 = \theta_2$. Hence $\langle\alpha_1\rangle = \langle\alpha_2\rangle$ by (5.4). We obtain that $\lfloor 1-a\alpha_1\rfloor = \lfloor 1-a\alpha_2\rfloor$ and (5.5) implies that $\lfloor 1-\alpha_1\rfloor = \lfloor 1-\alpha_2\rfloor$. Since $\alpha_i = \langle\alpha_i\rangle - \lfloor 1-\alpha_i\rfloor$, we get $\alpha_1 = \alpha_2$, as expected. This ends the proof. ∎

5.4. Efficient criteria for *q*-integrality and proof of Theorem 1.3. We are now ready to prove Theorem 1.3. The last missing ingredient is the following lemma.

Lemma 5.4. *Let* $\mathbf{r} = ((r_1, s_1), \dots, (r_v, s_v))$ *and* $\mathbf{t} = ((t_1, u_1), \dots, (t_w, u_w))$ *be vectors with integer coordinates such that, for all* (i, j), $s_i u_j \neq 0$ *and the ratios* r_i/s_i *and* t_j/u_j *do not belong to* $\mathbb{Z}_{\leq 0}$. *Then the following two assertions are equivalent:*

(i) *For all but finitely many* b, $\Delta_b^{\mathbf{r},\mathbf{t}}$ *is non-negative on* $\mathbb{R}_{\geq 0}$.

(ii) *For every* $b \in \{1, \dots, d_{\mathbf{r},\mathbf{t}}\}$ *and all* $x \in \mathbb{R}$, *we have* $\Xi_{\mathbf{r},\mathbf{t}}(b, x) \geq 0$.

Proof. We write Δ_b and $\Xi(b, \cdot)$ as respective shorthands for $\Delta_b^{\mathbf{r},\mathbf{t}}$ and $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$.

If b is large enough, then we infer from (5.2) that Δ_b is a step function whose jumps on $[0, 1]$ are precisely located at rationals of the form

$$\gamma(r, s, k) := \frac{D_{b/c}(\alpha) + k}{c} + \frac{\lfloor 1 - \alpha \rfloor}{b},$$

where (r, s) belongs either to $\mathbf{r}$ or to $\mathbf{t}$, $\alpha = r/s$, $c = \gcd(r, s, b)$ and $k \in \{0, \dots, c - 1\}$. More precisely, Δ_b has a jump of positive amplitude at each element of the multiset

$$\mathfrak{J}_b^+ := \left\{\left\{ \frac{D_{b/c_i}(\alpha_i) + k}{c_i} + \frac{\lfloor 1 - \alpha_i \rfloor}{b} : i \in V_b,\ 0 \leq k \leq c_i - 1 \right\}\right\}.$$

The amplitude of such a jump is equal to the multiplicity of the corresponding element in $\mathfrak{J}_b^+$. Similarly, Δ_b has a jump of negative amplitude at each element of the multiset

$$\mathfrak{J}_b^- := \left\{\left\{ \frac{D_{b/d_j}(\beta_j) + \ell}{d_j} + \frac{\lfloor 1 - \beta_j \rfloor}{b} : j \in W_b,\ 0 \leq \ell \leq d_j - 1 \right\}\right\},$$

and the amplitude of such a jump is equal to the multiplicity of the corresponding element in $\mathfrak{J}_b^-$. By Lemma 3.7, the supports of these multisets are included in $(0, 1]$. Let

$$0 < \gamma_1 < \cdots < \gamma_\mu \leq 1$$

denote the elements of the support of the multiset $\mathfrak{J}_b := \mathfrak{J}_b^+ \cup \mathfrak{J}_b^-$. We let m_i^+ (resp. m_i^-) denote the multiplicity of γ_i in $\mathfrak{J}_b^+$ (resp. in $\mathfrak{J}_b^-$), and we set $m_i := m_i^+ - m_i^-$. Let $x \in [0, 1]$ and set $\nu := \sup\{i \in \{1, \dots, \mu\} : \gamma_i \leq x\}$ with the convention $\sup(\emptyset) = -\infty$. Then, setting $\gamma_{-\infty} := 0$, we obtain

$$\Delta_b(x) = \Delta_b(\gamma_\nu) = \begin{cases} m_1 + \cdots + m_\nu & \text{if } \nu \geq 1, \\ 0 & \text{if } \nu = -\infty. \end{cases}$$

On the other hand, let $\underline{b}$ denote the unique representative of b in $\{1, \dots, d_{\mathbf{r},\mathbf{t}}\}$ modulo $d_{\mathbf{r},\mathbf{t}}$ and let us consider the multisets

$$\mathcal{J}_{\underline{b}}^+ := \left\{\left\{ \frac{\langle e_i \alpha_i \rangle + k}{c_i} - \lfloor 1 - a\alpha_i \rfloor : i \in V_{\underline{b}},\ 0 \leq k \leq c_i - 1 \right\}\right\},$$
$$\mathcal{J}_{\underline{b}}^- := \left\{\left\{ \frac{\langle f_j \beta_j \rangle + \ell}{d_j} - \lfloor 1 - a\beta_j \rfloor : j \in W_{\underline{b}},\ 0 \leq \ell \leq d_j - 1 \right\}\right\}.$$

By Lemma 5.2, the support of $\mathcal{J}_{\underline{b}} := \mathcal{J}_{\underline{b}}^+ \cup \mathcal{J}_{\underline{b}}^-$ also has cardinality μ. Let

$$\Gamma_1 \prec \cdots \prec \Gamma_\mu \tag{5.6}$$

denote the elements of the support of $\mathcal{J}_{\underline{b}}$ (in Christol order). Furthermore, Lemma 5.2 implies that Γ_i also has multiplicity m_i^+ in $\mathcal{J}_{\underline{b}}^+$ and m_i^- in $\mathcal{J}_{\underline{b}}^-$. Let $x \in \mathbb{R}$ and set $\nu := \sup\{i \in \{1, \dots, \mu\} : \Gamma_i \preceq x\}$. Then, setting $\Gamma_{-\infty} := 0$, we obtain

$$\Xi(\underline{b}, x) = \Xi(\underline{b}, \Gamma_\nu) = \begin{cases} m_1 + \cdots + m_\nu & \text{if } \nu \geq 1, \\ 0 & \text{if } \nu = -\infty. \end{cases}$$

We deduce that

$$\begin{aligned} \Delta_b([0,1]) &= \{0, \Delta_b(\gamma_1), \dots, \Delta_b(\gamma_\mu)\} \\ &= \{0, \Xi(\underline{b}, \Gamma_1), \dots, \Xi(\underline{b}, \Gamma_\mu)\} = \Xi(\underline{b}, \mathbb{R}). \end{aligned} \tag{5.7}$$

This shows that assertion (ii) is equivalent to Δ_b being non-negative on $[0,1]$ for all b large enough. On the other hand, the identity $\Delta_b(x+k) = \Delta_b(x) + k\Delta_b(1)$ proved for $k \in \mathbb{Z}$ in Lemma 4.3 shows that Δ_b is non-negative on $[0,1]$ if and only if it is non-negative on $\mathbb{R}_{\geq 0}$. Finally, we see that assertions (i) and (ii) are equivalent, which ends the proof. ■

Remark 5.5. We infer from (5.7) that the step function $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$ is non-negative on $\mathbb{R}$ if and only if $\Xi_{\mathbf{r},\mathbf{t}}(b, \Gamma_i) \geq 0$ for all i, $1 \leq i \leq \mu$. Furthermore, since the Γ_i's are given by (5.6) explicitly, one can easily compute $\Xi_{\mathbf{r},\mathbf{t}}(b, \Gamma_i)$.

We first deduce from Proposition 4.1 and Lemma 5.4 the following result.

Theorem 5.6. *Assume that $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is a well-defined sequence which is not eventually zero. Then the following two assertions are equivalent:*

(i) *There exists $C(q) \in \mathbb{Z}[q] \setminus \{0\}$ such that $C(q)^n Q_{\mathbf{r},\mathbf{t}}(q;n) \in \mathbb{Z}[q^{-1}, q]$ for every $n \geq 0$.*
(ii) *For every $b \in \{1, \dots, d_{\mathbf{r},\mathbf{t}}\}$ and all $x \in \mathbb{R}$, we have $\Xi_{\mathbf{r},\mathbf{t}}(b, x) \geq 0$.*

Finally, we can complete the proof of our main q-integrality criterion.

Proof of Theorem 1.3. The result is a straightforward consequence of Corollary 4.2 and Lemma 5.4. ■

As discussed in Section 4.1, efficient criteria for the q-integrality of the sequences $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\leq 0}$ and $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\in\mathbb{Z}}$ can also be derived from Theorems 1.3 and 5.6.

6. Examples and applications. In this last section, we give an overview of the computation of Christol step functions through some classical examples.

6.1. General considerations. We keep the general notation of this paper. For every $b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$, we have defined in Section 5.1 the step function $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$. Using the notation used in the proof of Lemma 5.4, we obtain

$$\Xi_{\mathbf{r},\mathbf{t}}(b, x) := \#\{\{\gamma \in \mathcal{J}_b^+ : \gamma \preceq x\}\} - \#\{\{\gamma \in \mathcal{J}_b^- : \gamma \preceq x\}\}.$$

When $b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$ is coprime to $d_{\mathbf{r},\mathbf{t}}$, the function $\Xi_{\mathbf{r},\mathbf{t}}(b, \cdot)$ is easier to compute since

$$\mathcal{J}_b^+ = \{\{a\alpha_1, \ldots, a\alpha_v\}\} \quad \text{and} \quad \mathcal{J}_b^- = \{\{a\beta_1, \ldots, a\beta_w\}\},$$

where a is the unique integer in $\{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$ satisfying $ab \equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$. Theorem A can then be rephrased as follows:

$$\begin{aligned}&(Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n))_{n\geq 0} \text{ is } N\text{-integral}\\ &\quad \iff \forall b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\} \text{ with } \gcd(b, d_{\mathbf{r},\mathbf{t}}) = 1,\ \forall x \in \mathbb{R}\ \Xi_{\mathbf{r},\mathbf{t}}(b, x) \geq 0.\end{aligned}$$

Starting with an N-integral hypergeometric sequence

$$\frac{(\alpha_1)_n \cdots (\alpha_v)_n}{(\beta_1)_n \cdots (\beta_w)_n}, \qquad n \geq 0,$$

and taking $\mathbf{r}$ and $\mathbf{t}$ such that

$$Q_{\mathbf{r},\mathbf{t}}(q; n) = \frac{(q^{r_1}; q^{s_1})_n \cdots (q^{r_v}; q^{s_v})_n}{(q^{t_1}; q^{u_1})_n \cdots (q^{t_w}; q^{u_w})_n},$$

with $r_i/s_i = \alpha_i$ and $t_j/u_j = \beta_j$, Lemma 5.2 ensures the existence of a constant $\mathfrak{c}_{\mathbf{r},\mathbf{t}}$ such that, for every integer b coprime to $d_{\mathbf{r},\mathbf{t}}$ and larger than $\mathfrak{c}_{\mathbf{r},\mathbf{t}}$, we have

$$v_{\phi_b}(Q_{\mathbf{r},\mathbf{t}}(q; n)) = \Delta_b^{\mathbf{r},\mathbf{t}}(n/b) \geq 0.$$

Indeed, for $b > \mathfrak{c}_{\mathbf{r},\mathbf{t}}$, Lemma 5.2 shows that the $\leq$-ordering of the jumps of $\Delta_b^{\mathbf{r},\mathbf{t}}$ on $[0, 1]$ is the same as the $\preceq$-ordering of the jumps of $\Xi_{\mathbf{r},\mathbf{t}}(\underline{b}, \cdot)$ on $\mathbb{R}$, where $\underline{b}$ is the unique representative in $\{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$ of b modulo $d_{\mathbf{r},\mathbf{t}}$. In particular, $\Delta_b^{\mathbf{r},\mathbf{t}}$ is non-negative on $\mathbb{R}_{\geq 0}$ as expected.

Hence the denominator of $Q_{\mathbf{r},\mathbf{t}}(q; n)$ could only contain cyclotomic polynomials $\phi_b(q)$ with $b \leq \mathfrak{c}_{\mathbf{r},\mathbf{t}}$ or b not coprime to $d_{\mathbf{r},\mathbf{t}}$. The situation with such numbers b is much more complicated and strongly depends on the gcd's of the pairs (r_i, s_i) and (t_j, u_j).

Let us first consider the case where $\gcd(r_i, s_i) = 1$ and $\gcd(t_j, u_j) = 1$ for all i and j. Let $b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$, let $\tilde{b}$ be the greatest divisor of b coprime to $d_{\mathbf{r},\mathbf{t}}$, and let a be the unique integer in $\{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$ satisfying $a\tilde{b} \equiv 1 \bmod d_{\mathbf{r},\mathbf{t}}$. Then, following the notation of Section 5.1, we find $c_i = d_j = 1$, so that

$$\begin{aligned}V_b &= \{1 \leq i \leq v : \gcd(s_i, b) = 1\},\\ W_b &= \{1 \leq j \leq w : \gcd(u_j, b) = 1\},\end{aligned}$$

which yields

$$\mathcal{J}_b^+ = \{\{\langle e_i\alpha_i\rangle - \lfloor 1-a\alpha_i\rfloor : i\in V_b\}\},$$
$$\mathcal{J}_b^- = \{\{\langle f_j\beta_j\rangle - \lfloor 1-a\beta_j\rfloor : j\in W_b\}\}.$$

Hence each "classical" jump occurring at $a\alpha_i$ (by this, we mean the jumps occurring when b is coprime to $d_{\mathbf{r},\mathbf{t}}$) either disappears because b is not coprime to s_i, or is replaced by a jump at $\langle e_i\alpha_i\rangle - \lfloor 1-a\alpha_i\rfloor$ when b is coprime to s_i. Even in this particular case, we already understand that the new step functions can behave in a very different way than the classical ones.

As an illustration, we consider the simple example

$$Q_{\mathbf{r},\mathbf{t}}(q;n) := \frac{(q;q^3)_n(q^2;q^3)_n}{(q;q^2)_n(q;q)_n},$$

which was introduced at the end of Section 1.1 and corresponds to $\mathbf{r} = ((1,3),(2,3))$ and $\mathbf{t} = ((1,2),(1,1))$. We have

$$\left(\frac{(1-q^2)(1-q)}{(1-q^3)^2}\right)^n Q_{\mathbf{r},\mathbf{t}}(q;n) \xrightarrow{q\to 1} \frac{(1/3)_n(2/3)_n}{(1/2)_n(1)_n},$$

the right-hand side being N-integral. This can be derived from (1.2). We find that $d_{\mathbf{r},\mathbf{t}} = 6$, and for $b=3$ we obtain $\tilde{b}=1$ and $a=1$. This yields $V_3=\emptyset$, $W_3=\{1,2\}$, and $f_1=f_2=1$. Hence $\mathcal{J}_3^+=\emptyset$ and $\mathcal{J}_3^- = \{\{1/2,1\}\}$, so that $\Xi(3,1/2)<0$. Thus, we deduce from Theorem 1.3 that the sequence $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is not q-integral.

On the other hand, we have

$$\frac{(q;q^3)_n(q^2;q^3)_n}{(q;q^2)_n(q;q)_n}\cdot\frac{(q^3;q^3)_n}{(q^2;q^2)_n} = \begin{bmatrix}3n\\2n\end{bmatrix}_q \in \mathbb{Z}[q], \tag{6.1}$$

which shows that the corresponding q-hypergeometric sequence is obviously q-integral. In order to understand the effect of the extra factors $(q^3;q^3)_n$ and $(q^2;q^2)_n$, we have to investigate the case where $\gcd(r_i,s_i)\neq 1$.

When $\gcd(r_i,s_i)\neq 1$, we possibly have $c_i=\gcd(r_i,s_i,b)\neq 1$. In this case, either $\gcd(s_i,b)\neq c_i$ and the "classical" jump at $a\alpha_i$ disappears, or there is an integer e_i satisfying $be_i\equiv c_i \bmod s_i$ and the jump at $a\alpha_i$ splits into c_i distinct jumps at

$$\frac{\langle e_i\alpha_i\rangle + k}{c_i} - \lfloor 1-a\alpha_i\rfloor, \qquad 0\leq k\leq c_i-1.$$

Let us now return to (6.1) and consider the case where $b=3$. Then, we find that $c_3=3$, $V_3=\{3\}$, and $e_3=1$. This yields jumps with amplitude $+1$ at all elements of the (multi)set $\mathcal{J}_3^+ = \{\{1/3,2/3,1\}\}$. On the other hand, we have $W_3=\{1,2,3\}$ and $f_1=f_2=f_3=1$, which yields jumps with amplitude -1 at all elements of the multiset $\mathcal{J}_3^- = \{\{1/2,1,1\}\}$. In the end, we get

$$\Gamma_1 = \tfrac{1}{3} \prec \Gamma_2 = \tfrac{1}{2} \prec \Gamma_3 = \tfrac{2}{3} \prec \Gamma_4 = 1, \tag{6.2}$$

with $m_1 = 1$, $m_2 = -1$, $m_3 = 1$, and $m_4 = -1$. It follows that the step function $\Xi(3, \cdot)$ is non-negative on $\mathbb{R}$, as expected.

6.2. q-Factorial ratios. Let us recall that $[n]_q = (1 - q^n)/(1 - q)$, so that

$$[n]_q = \prod_{b \geq 2,\, b|n} \phi_b(q)$$

and

$$[n]!_q := \prod_{i=1}^{n} \frac{1 - q^i}{1 - q} = \prod_{b \geq 2} \phi_b(q)^{\lfloor n/b \rfloor}. \tag{6.3}$$

Given two vectors $e := (e_1, \ldots, e_v)$ and $f := (f_1, \ldots, f_w)$ whose coordinates are positive integers, we define as in [22] the q-analog of the factorial ratio $Q_{e,f}(n)$ as

$$Q_{e,f}(q; n) := \frac{[e_1 n]!_q \cdots [e_v n]!_q}{[f_1 n]!_q \cdots [f_w n]!_q} \cdot$$

We deduce from (6.3) that

$$Q_{e,f}(q; n) = \prod_{b \geq 2} \phi_b(q)^{\Delta_{e,f}(n/b)},$$

where

$$\Delta_{e,f}(x) = \sum_{i=1}^{v} \lfloor e_i x \rfloor - \sum_{j=1}^{w} \lfloor f_j x \rfloor$$

is the classical Landau function, as defined in (1.1). We easily see that $Q_{e,f}(q; n)$ is q-integral if and only if $\Delta_{e,f}$ is non-negative on $[0, 1]$. Note that these properties are also equivalent to $Q_{e,f}(q; n) \in \mathbb{Z}[q]$ (see also [22] where a positivity conjecture for the coefficients of these polynomials is proposed). It is therefore much more efficient to work with $\Delta_{e,f}$ than to compute the corresponding Christol functions.

The example given in (6.1) corresponds to $e = (3)$ and $f = (2, 1)$, so that

$$\Delta_{e,f}(x) = \lfloor 3x \rfloor - \lfloor 2x \rfloor - \lfloor x \rfloor.$$

On $[0, 1]$, this step function has jumps with positive amplitude $+1$ at $1/3$ and $2/3$, and jumps with negative amplitude -1 at $1/2$ and 1. As expected, we retrieve the same ordering as in (6.2) for the jumps of $\Xi(3, \cdot)$.

6.3. A famous non-factorial example. When introducing his step functions in [11], Christol was motivated by the following question: is it true that an N-integral hypergeometric series is the diagonal of a rational fraction in several variables? The hypergeometric sequence

$$\frac{(1/9)_n(4/9)_n(5/9)_n}{(1/3)_n(1)_n^2}, \qquad n \geq 0, \tag{6.4}$$

is one of the simplest examples of an N-integral hypergeometric sequence for which the question is still open (although recent progress in this direction has been made in [1, 8]).

In this case, the six Christol functions associated with each b coprime to 9 are non-negative on $\mathbb{R}$. By Theorem A, this ensures that this hypergeometric sequence is N-integral. A precise formula for the smallest positive integer N_0 is given in [14, Theorem 4]: here we get $N_0 = 9^3$.

As already discussed, a natural q-analog of (6.4) can be defined as

$$\left(\frac{(1-q^3)(1-q)^2}{(1-q^9)^3}\right)^n \frac{(q;q^9)_n(q^4;q^9)_n(q^5;q^9)_n}{(q;q^3)_n(q;q)_n^2}, \qquad n \geq 0.$$

The q-integrality of this sequence is equivalent to the one of the q-hypergeometric sequence $Q_{\mathbf{r},\mathbf{t}}(q;n)$, where $\mathbf{r} = ((1,9),(4,9),(5,9))$ and $\mathbf{t} = ((1,3),(1,1),(1,1))$.

It remains to consider the Christol functions associated with $b \in \{3,6,9\}$. For $b = 3$, we have $\gcd(9,b) = 3 \neq 1$ so that $\mathcal{J}_3^+ = \emptyset$. But due to the factors $(q;q)_n^2$ in the denominator, we obtain $\mathcal{J}_3^- = \{\{1,1\}\}$, so that $\Xi_{\mathbf{r},\mathbf{t}}(b,1) < 0$. We deduce from Theorem 1.3 that the sequence $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is not q-integral. In this example, all the “classical” jumps with positive amplitude have disappeared for $b = 3$.

In fact, we can retrieve q-integrality by adding a factor $(q^9;q^9)_n$ to the numerator and a factor $(q;q)_n$ to the denominator. This leads to the slightly modified q-analog:

$$\left(\frac{(1-q^3)(1-q)^3}{(1-q^9)^4}\right)^n \frac{(q;q^9)_n(q^4;q^9)_n(q^5;q^9)_n(q^9;q^9)_n}{(q;q^3)_n(q;q)_n^3}, \qquad n \geq 0.$$

With this new choice of parameters $\mathbf{r}'$ and $\mathbf{t}'$, the functions $\Xi_{\mathbf{r}',\mathbf{t}'}(b,\cdot)$ for b coprime to 9 remains unchanged. However, for b in $\{3,6,9\}$, one finds that V_b is no longer empty. A computation shows that $V_b = \{4\}$, $W_b = \{2,3,4\}$, $\mathcal{J}_3^- = \mathcal{J}_9^- = \{\{1,1,1\}\}$, $\mathcal{J}_6^- = \{\{5,5,5\}\}$, while

$$\mathcal{J}_3^+ = \{\{\tfrac{1}{3}, \tfrac{2}{3}, 1\}\}, \quad \mathcal{J}_6^+ = \{\{\tfrac{1}{3}+4, \tfrac{2}{3}+4, 5\}\}, \quad \mathcal{J}_9^+ = \{\{\tfrac{1}{9}, \tfrac{2}{9}, \ldots, \tfrac{8}{9}, 1\}\}.$$

In all cases, $\Xi_{\mathbf{r}',\mathbf{t}'}(b,\cdot)$ is now non-negative on $\mathbb{R}$ and we infer from Theorem 5.6 that the sequence $(Q_{\mathbf{r}',\mathbf{t}'}(q;n))_{n\geq 0}$ is q-integral.

Finally, we consider a third q-analog of the hypergeometric sequence (6.4), which we define as

$$\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;n) = \frac{(q^{1/9};q)_n(q^{4/9};q)_n(q^{5/9};q)_n}{(q^{1/3};q)_n(q;q)_n^2}, \qquad n \geq 0. \tag{6.5}$$

As already discussed, the $q^{1/9}$-integrality of $(\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q;n))_{n\geq 0}$ is equivalent to the q-integrality of the sequence

$$\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q^9;n)=\frac{(q;q^9)_n(q^4;q^9)_n(q^5;q^9)_n}{(q^3;q^9)_n(q^9;q^9)_n^2},\qquad n\geq 0.$$

Furthermore, we have $\widetilde{Q}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q^9;n)=Q_{\mathbf{r},\mathbf{t}}(q;n)$ for a suitable choice of vectors $\mathbf{r}$ and $\mathbf{t}$. As previously, a computation shows that for $b=3$, we have $\mathcal{J}_3^+=\emptyset$, while $1\in\mathcal{J}_3^-$, so that $\Xi_{\mathbf{r},\mathbf{t}}(3,1)<0$. We deduce from Theorem 5.6 that $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is not q-integral. Hence the sequence defined in (6.5) is not $q^{1/9}$-integral.

We observe that, in this case, we cannot use the same trick as before. Indeed, multiplying $\widetilde{Q}_{\mathbf{r},\mathbf{t}}(q^9;n)$ by $(q^9;q^9)_n/(q;q)_n$ amounts to multiplying (6.5) by $(q;q)_n/(q^{1/9};q^{1/9})_n$ which does not correspond to any choice of parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

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References

[1] Y. Abdelaziz, C. Koutschan, and J.-M. Maillard, *On Christol's conjecture*, J. Phys. A 53 (2020), art. 205201, 16 pp.

[2] B. Adamczewski, J. P. Bell, and É. Delaygue, *Algebraic independence of G-functions and congruences "à la Lucas"*, Ann. Sci. École Norm. Sup. 52 (2019), 515–559.

[3] A. Adolphson and S. Sperber, *On the integrality of hypergeometric series whose coefficients are factorial ratios*, Acta Arith. 200 (2021), 39–59.

[4] A. Adolphson and S. Sperber, *On integrality properties of hypergeometric series*, Funct. Approx. Comment. Math. 65 (2021), 7–31.

[5] V. V. Batyrev and D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersection in toric varieties*, Comm. Math. Phys. 168 (1995), 493–533.

[6] F. Beukers and G. Heckman, *Monodromy for the hypergeometric functions* ${}_nF_{n-1}$, Invent. Math. 95 (1989), 325–354.

[7] J. W. Bober, *Factorial ratios, hypergeometric series, and a family of step functions*, J. London Math. Soc. 79 (2009), 422–444.

[8] A. Bostan and S. Yurkevich, *On a class of hypergeometric diagonals*, Proc. Amer. Math. Soc. 150 (2022), 1071–1087.

[9] P. L. Chebyshev, *Mémoire sur les nombres premiers*, J. Math. Pures Appl. 17 (1852), 366–390.

[10] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Phys. Lett. B 359 (1991), 21–74.

[11] G. Christol, *Fonctions hypergéométriques bornées*, Groupe de travail d'analyse ultramétrique, 14 (1986-1987), exp. 8, 1–16.

[12] É. Delaygue, *Critère pour l'intégralité des coefficients de Taylor des applications miroir*, J. Reine Angew. Math. 662 (2012), 205–252.

[13] É. Delaygue, *Arithmetic properties of Apéry-like numbers*, Compos. Math. 154 (2018), 249–274.
[14] É. Delaygue, T. Rivoal, and J. Roques, *On Dwork's p-adic formal congruences theorem and hypergeometric mirror maps*, Mem. Amer. Math. Soc. 246 (2017), no. 1163, v+94 pp.
[15] B. Dwork, *On p-adic differential equations IV. Generalized hypergeometric functions as p-adic analytic functions in one variable*, Ann. Sci. École Norm. Sup. 6 (1973), 295–315.
[16] J. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia Math. Appl. 96, Cambridge Univ. Press, Cambridge, 2004.
[17] E. Heine, *Über die Reihe* $1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}x^2+\dots$ *(Aus einem Schreiben an Lejeune Dirichlet)*, J. Reine Angew. Math. 32 (1846), 210–212.
[18] N. Katz, *Algebraic solutions of differential equations*, Invent. Math. 18 (1972), 1–118.
[19] E. Landau, *Sur les conditions de divisibilité d'un produit de factorielles par un autre*, in: Collected Works, I, Thales-Verlag, 1985, p. 116.
[20] F. Rodriguez-Villegas, *Integral ratios of factorials and algebraic hypergeometric functions*, arXiv:math/0701.1362v1 (2007).
[21] K. Soundararajan, *Integral factorial ratios*, Duke Math. J. 171 (2022), 633–672.
[22] S. O. Warnaar and W. Zudilin, *A q-rious positivity*, Aequationes Math. 81 (2011), 177–183.

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